GROMOV-WITTEN THEORY AND NOETHER-LEFSCHETZ THEORY

D. MAULIK AND R. PANDHARIPANDE

Abstract. Noether-Lefschetz divisors in the moduli of $K3$ surfaces are the loci corresponding to Picard rank at least 2. We relate the degrees of the Noether-Lefschetz divisors in 1-parameter families of $K3$ surfaces to the Gromov-Witten theory of the 3-fold total space. The reduced $K3$ theory and the Yau-Zaslow formula play an important role. We use results of Borcherds and Kudla-Millson for $O(2,19)$ lattices to determine the Noether-Lefschetz degrees in classical families of $K3$ surfaces of degrees 2, 4, 6 and 8. For the quartic $K3$ surfaces, the Noether-Lefschetz degrees are proven to be the Fourier coefficients of an explicitly computed modular form of weight 21/2 and level 8. The interplay with mirror symmetry is discussed. We close with a conjecture on the Picard ranks of moduli spaces of $K3$ surfaces.

Contents

0. Introduction

0.1. **K3 families.** Let C be a nonsingular complete curve, and let

$$\pi : X \to C$$

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be a 1-parameter family of nonsingular quasi-polarized $K3$ surfaces. Let $L \in \mathrm{Pic}(X)$ denote the quasi-polarization of degree

$$\int_{K3} L^2 = l \in 2\mathbb{Z}^{>0}.$$

The family π yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_l,$$

to the 19 dimensional moduli space of quasi-polarized $K3$ surfaces of degree l. A review of the definitions can be found in Section 1.

0.2. **Noether-Lefschetz numbers.** Noether-Lefschetz numbers are defined by the intersection of $\iota_\pi(C)$ with Noether-Lefschetz divisors in $\mathcal{M}_l$. Noether-Lefschetz divisors can be described via Picard lattices or Picard classes. We briefly review the two approaches.

Let $(\mathbb{L}, v)$ be a rank 2 integral lattice with an even symmetric bilinear form

$$\langle , \rangle : \mathbb{L} \times \mathbb{L} \to \mathbb{Z}$$

and a distinguished primitive vector $v \in \mathbb{L}$ satisfying

$$\langle v, v \rangle = l.$$

The invariants of $(\mathbb{L}, v)$ are the discriminant $\triangle \in \mathbb{Z}$ and the coset

$$\delta \in \left(\frac{\mathbb{Z}}{l\mathbb{Z}}\right) / \pm .$$

If the data are presented as

$$\mathbb{L}_{h,d} = \begin{pmatrix} l & d \\ d & 2h-2 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then the discriminant is

$$\triangle_l(h,d) = -\det \begin{vmatrix} l & d \\ d & 2h-2 \end{vmatrix} = d^2 - 2lh + 2l$$

and the coset is

$$\delta = d \bmod l \quad \in \left(\frac{\mathbb{Z}}{l\mathbb{Z}}\right) / \pm .$$

Two lattices $(\mathbb{L}_{h,d}, v)$ and $(\mathbb{L}_{h',d'}, v')$ are equivalent if and only if

$$\triangle_l(h,d) = \triangle_l(h',d') \ \text{ and } \ \delta_{h,d} = \delta_{h',d'}.$$

However, not all pairs $(\triangle, \delta)$ are realized.

The first type of Noether-Lefschetz divisor is defined by specifying a Picard lattice. Let

$$P_{\triangle,\delta} \subset \mathcal{M}_l$$

be the closure of the locus of quasi-polarized $K3$ surfaces (S, L) of degree l for which $(\mathrm{Pic}(S), L)$ is of rank 2 with discriminant $\triangle$ and coset δ. By the Hodge index theorem, $P_{\triangle,\delta}$ is empty unless $\triangle > 0$.

The second type of Noether-Lefschetz divisor is defined by specifying a Picard class. In case $\triangle_l(h, d) > 0$, let

$$D_{h,d} \subset \mathcal{M}_l$$

have support on the locus of quasi-polarized $K3$ surfaces (S, L) for which there exists a class $\beta \in \mathrm{Pic}(S)$ satisfying

$$\int_S \beta^2 = 2h - 2 \quad \text{and} \quad \int_S \beta \cdot L = d.$$

More precisely, $D_{h,d}$ is the weighted sum

$$D_{h,d} = \sum_{\triangle,\delta} \mu(h, d \mid \triangle, \delta) \cdot [P_{\triangle,\delta}] \tag{1}$$

where the multiplicity

$$\mu(h, d \mid \triangle, \delta) \in \{0, 1, 2\}$$

is defined to be the number of elements β of the lattice $(\mathbb{L}, v)$ associated to $(\triangle, \delta)$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \langle \beta, v \rangle = d. \tag{2}$$

If no lattice corresponds to $(\triangle, \delta)$, the multiplicity $\mu(h, d \mid \triangle, \delta)$ vanishes and $P_{\triangle,\delta}$ is empty. If the multiplicity is nonzero, then

$$\triangle \mid \triangle_l (h, d).$$

Hence, the sum on the right of (1) has only finitely many terms.

As relation (1) is easily seen to be triangular, the divisors $P_{\triangle,\delta}$ and $D_{h,d}$ are essentially equivalent. However, the divisors $D_{h,d}$ will be seen to have better formal properties.

A natural approach to studying the divisors $D_{h,d}$ is via intersections with test curves. In case $\triangle_l(h, d) > 0$, the Noether-Lefschetz number $NL^\pi_{h,d}$ is the classical intersection product

$$NL^\pi_{h,d} = \int_C \iota^*_\pi [D_{h,d}]. \tag{3}$$

If $\triangle_l(h, d) < 0$, the divisor $D_{h,d}$ vanishes by the Hodge index theorem. A definition of $NL^\pi_{h,d}$ for all values $\triangle_l(h, d) \geq 0$ is given by classical intersection in the period domain for $K3$ surfaces in Section 1.

The divisibility of a nonzero element β of a lattice is the maximal positive integer m dividing β. Refined divisors $D_{m,h,d}$ are defined by

$$D_{m,h,d} = \sum_{\triangle,\delta} \mu(m,h,d\,|\triangle,\delta) \cdot [P_{\triangle,\delta}]$$

where the multiplicity

$$\mu(m,h,d\,|\triangle,\delta) \in \{0,1,2\}$$

is the number of elements β of divisibility m of the lattice $(\mathbb{L}, v)$ associated to $(\triangle,\delta)$ satisfying (2). Refined Noether-Lefschetz number are defined by

$$NL^{\pi}_{m,h,d} = \int_C \iota_\pi^*[D_{m,h,d}].$$

0.3. **Invariants.** We will study three types of invariants associated to a 1-parameter family π of quasi-polarized $K3$ surfaces in case the total space X is nonsingular:

(i) the Noether-Lefschetz numbers of π,
(ii) the Gromov-Witten invariants of X,
(iii) the reduced Gromov-Witten invariants of the $K3$ fibers.

The Noether-Lefschetz numbers (i) are classical intersection products while the Gromov-Witten invariants (ii)-(iii) are quantum in origin.

The Gromov-Witten invariants (ii) of the 3-fold X and the reduced Gromov-Witten invariants (iii) of a $K3$ surface are defined via integration against virtual classes of moduli spaces of stable maps. We view both of these Gromov-Witten theories in terms of the associated BPS state counts defined by Gopakumar and Vafa [17, 18].

Let $n^X_{g,d}$ denote the Gopakumar-Vafa invariant of X of genus g for π-vertical curve classes of degree d with respect to L. Let $r_{g,m,h}$ denote the Gopakumar-Vafa reduced $K3$ invariant of genus g and curve class $\beta \in H_2(K3,\mathbb{Z})$ of divisibility m satisfying

$$\int_{K3} \beta^2 = 2h-2.$$

A review of these quantum invariants is presented in Section 2.

A geometric result intertwining the invariants (i)-(iii) is derived in Section 3 by a comparison of the reduced and usual deformation theories of maps of curves to the $K3$ fibers of π.

Theorem 1. For $d > 0$,

$$n^X_{g,d} = \sum_h \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^{\pi}_{m,h,d}.$$

Theorem 1 is the main geometric result of the paper. The proof is given in Section 3.

0.4. **Applications.** Since Theorem 1 relates three distinct geometric invariants, the result can be effectively used in several directions.

An application for studying reduced invariants of $K3$ surfaces is given in [25]. A central conjecture discussed in Section 2.3 is the *independence*[1] of $r_{g,m,h}$ on m. In genus 0, the independence is the non-primitive Yau-Zaslow conjecture proven in [25] as a consequence of Theorem 1.

The approach taken there is the following. For a specific 1-parameter family of $K3$ surfaces, known in the physics literature as the STU model, the BPS states $n_{0,d}^{STU}$ are known by proven mirror transformations and the Noether-Lefschetz numbers $NL_{m,h,d}^{STU}$ can by exactly determined. Theorem 1 is then used in [25] to solve for $r_{0,m,h}$:

$$r_{0,m,h} = r_{0,1,h}, \qquad \sum_{h \geq 0} r_{0,1,h} = \prod_{n \geq} \frac{1}{(1-q^n)^{24}} \ .$$

The genus 1 results

$$r_{1,m,h} = r_{1,1,h} = -\frac{h}{12}\ r_{0,1,h}$$

are an easy consequence, see Section 2.3. We write $r_{g,m,h} = r_{g,h}$ independent of m for $g = 0, 1$.

Using [25], the genus 0 and 1 specialization takes a much simpler form.

Corollary 1. For $g \leq 1$ and $d > 0$,

$$n_{g,d}^X = \sum_{h=g}^{\infty} r_{g,h} \cdot NL_{h,d}^{\pi}.$$

By Corollary 1, the Gromov-Witten invariants $n_{g,d}^X$ are completely determined by the Noether-Lefschetz numbers of π for any 1-parameter family of quasi-polarized $K3$ surfaces. The result may be viewed as giving a fully classical interpretation of the Gromov-Witten invariants of X in π-vertical classes.

Theorem 1 can also be used to constrain the Noether-Lefschetz degrees themselves. An important approach to the Noether-Lefschetz numbers (already used in the STU calculation) is via results of Borcherds [6] and Kudla-Millson [27]. The Noether-Lefschetz numbers of π are

[1] If m^2 does not divide $2h-2$, then $r_{g,m,h} = 0$. The independence is conjectured only when m^2 divides $2h-2$. When we write $r_{g,m,h}$, the divisibility condition is understood to hold.

proven to be the Fourier coefficients of a vector-valued modular form.[2] For several classical families of $K3$ surfaces, Corollary 1 in genus 0 provides an alternative method of calculating the Noether-Lefschetz numbers via the invariants $n^X_{0,d}$. Together, we obtain a remarkable sequence of identities intertwining hypergeometric series from mirror transformations (calculating $n^X_{0,d}$) and modular forms. The Harvey-Moore identity [20] for the STU model is a special case.

As a basic example, we provide a complete calculation of the Noether-Lefschetz numbers for the family of $K3$ surfaces determined by a Lefschetz pencil of quartics in $\mathbb{P}^3$. The required mirror symmetry calculations (iii) for the quartic pencil have long been established rigorously [15, 16]. We give the derivation of the Noether-Lefschetz numbers via Gromov-Witten calculations in Section 5. The resulting hypergeometric-modular identity follows immediately in Section 5.5. A second approach to calculating Noether-Lefschetz numbers directly via more sophisticated modular form techniques is explained for quartics and several other classical families in Section 6.

Once the Noether-Lefschetz numbers are calculated for the 1-parameter family π, Corollary 1 yields the genus 1 Gromov-Witten invariants of X in π-vertical classes. There are very few methods for the exact calculation of genus 1 invariants in Calabi-Yau geometries.[3] Corollary 1 provides a new class of complete solutions.

0.5. **Heterotic duality.** In rather different terms, approach (i)-(iii) was pursued in the string theoretic work of Klemm, Kreuzer, Riegler, and Scheidegger [24] with the goal of calculating the BPS counts $n^X_{g,d}$ from the genus 0 values $n^X_{0,d}$. Heterotic duality was used in [24] for (i) since the connection to the intersection theory of the Noether-Lefschetz divisors

$$D_{h,d} \subset \mathcal{M}_l$$

and the work of Borcherds was not made. The perspective of [24] can be turned upside down by using Gromov-Witten theory to calculate the Noether-Lefschetz numbers. On the other hand, modularity allows the calculations of [24] to be pursued in much greater generality.

In fact, the back and forth here between heterotic duality and mathematical results is older. Borcherds' paper on automorphic functions [5] which underlies [6] was motivated in part by the work of Harvey

[2]While the paper [6, 27] have considerable overlap, we will follow the point of view of Borcherds.

[3]See [49] for a different mathematical approach to genus 1 invariants for complete intersections.

and Moore [20, 21] on heterotic duality. The first higher genus results for $K3$ fibrations were by Mariño and Moore [36].

Finally, we mention the circle ideas here can be considered for interesting isotrivial families of $K3$ surfaces with double Enriques fibers [26, 37]. While heterotic duality arguments apply there, Borcherds' result does not directly apply.

0.6. **Modular forms.** Let A and B be modular forms of weight $1/2$ and level 8,

$$A = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{8}}, \quad B = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{8}}.$$

Let Θ be the modular form of weight $21/2$ and level 8 defined by

$$\begin{aligned} 2^{22}\Theta \;\; = \;\; & 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 \\ & -20007A^{16}B^5 - 169092A^{15}B^6 - 120636A^{14}B^7 \\ & -621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} \\ & -346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} \\ & -361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} \\ & -4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}. \end{aligned}$$

We can expand Θ as a series in $q^{\frac{1}{8}}$,

$$\Theta = -1 + 108q + 320q^{\frac{9}{8}} + 5016q^{\frac{3}{2}} + \ldots.$$

The modular form Θ was first found in calculations of [24].

Let π be the family of quasi-polarized $K3$ surfaces determined by a Lefschetz pencil of quartics in $\mathbb{P}^4$. Let $\Theta[m]$ denote the coefficient of q^m in Θ.

Theorem 2. *The Noether-Lefschetz numbers of the quartic pencil π are coefficients of Θ,*

$$NL^{\pi}_{h,d} = \Theta\left[\frac{\triangle_4(h,d)}{8}\right].$$

0.7. **Classical quartic geometry.** Let V be a 4-dimensional $\mathbb{C}$-vector space. A quartic hypersurface in $\mathbb{P}(V)$ is determined by an element of $\mathbb{P}(\mathrm{Sym}^4 V^*)$. Let

$$\mathcal{U} \subset \mathbb{P}(\mathrm{Sym}^4 V^*)$$

be the Zariski open set of nonsingular quartic hypersurfaces. Since $[S] \in \mathcal{U}$ corresponds to a polarized $K3$ surface of degree 4, we obtain a canonical morphism

$$\phi : \mathcal{U} \to \mathcal{M}_4.$$

If $\triangle_4(h,d) > 0$, the pull-back

$$\mathcal{D}_{h,d} = \phi^{-1}(D_{h,d}) \subset \mathcal{U}$$

is a closed subvariety of pure codimension 1. As a Corollary of Theorem 2, we obtain a complete calculation of the degrees of the hypersurfaces

$$\overline{\mathcal{D}}_{h,d} \subset \mathbb{P}(\mathrm{Sym}^4 V^*).$$

Corollary 2. *If $\triangle_4(h,d) > 0$, the degree of $\overline{\mathcal{D}}_{h,d}$ is*

$$\deg(\overline{\mathcal{D}}_{h,d}) = \Theta\left[\frac{\triangle_4(h,d)}{8}\right] - \Psi\left[\frac{\triangle_4(h,d)}{8}\right]$$

where the correction term is

$$\Psi = 108 \sum_{n>0} q^{n^2}.$$

The correction term, obtained from the contribution of the nodal quartics, is explained in Section 5.6. Formulas for the degrees of

$$\overline{\phi^{-1}(P_{\triangle,\delta})} \subset \mathbb{P}(\mathrm{Sym}^4 V^*)$$

are easily obtained from (1) and a parallel nodal analysis. While Corollary 2 answers a classical question about the Hodge theory of quartic $K3$ surfaces, the method of proof is modern.

0.8. **Outline.** In Section 1, we give a precise definition of Noether-Lefschetz numbers and establish several elementary properties. The definitions of BPS invariants for 3-folds and reduced Gromov-Witten invariants of $K3$ surfaces are recalled in Section 2. Two central conjectures about the reduced theory of $K3$ surfaces are stated in Section 2.3. The proof of Theorem 1 is presented in Section 3.

We review of the work of Borcherds on Heegner divisors and explain the application to families of $K3$ surfaces in Section 4. The results are applied with Theorem 1 to prove Theorem 2 via mirror symmetry calculations in Section 5. A direct approach to Noether-Lefschetz degrees for classical familes of $K3$ surfaces of degrees $2, 4, 6$, and 8 is given in Section 6 via a deeper study of vector-valued modular forms. Finally, in Section 7, we state a conjecture regarding Picard ranks of moduli spaces of $K3$ surfaces of degree l .

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1. Noether-Lefschetz numbers

1.1. **Picard lattice.** Let S be a $K3$ surface. The second cohomology of S is a rank 22 lattice with intersection form

$$H^2(S,\mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \tag{4}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (4) is even.

The *divisibility* of $\beta \in H^2(S,\mathbb{Z})$ is the maximal positive integer dividing β. If the divisibility is 1, β is *primitive*. Elements with equal divisibility and norm are equivalent up to orthogonal transformation of $H^2(S,\mathbb{Z})$.

The Hodge decomposition of the second cohomology of S has dimensions $(1,20,1)$,

$$H^2(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^{2,0}(S,\mathbb{C}) \oplus H^{1,1}(S,\mathbb{C}) \oplus H^{0,2}(S,\mathbb{C}).$$

The *Picard lattice* of S is

$$\mathrm{Pic}(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S,\mathbb{C}).$$

1.2. **Quasi-polarization.** A *quasi-polarization* on S is a line bundle L with primitive Chern class $c_1(L) \in H^2(S,\mathbb{Z})$ satisfying

$$\int_S L^2 > 0 \quad \text{and} \quad \int_S L \cdot [C] \geq 0$$

for every curve $C \subset S$. A sufficiently high tensor power L^n of a quasi-polarization is base point free and determines a birational morphism

$$S \to \widetilde{S}$$

contracting A-D-E configurations of (-2)-curves on S. Hence, every quasi-polarized $K3$ surface (S, L) is algebraic.

Let X be a compact 3-dimensional complex manifold equipped with a holomorphic line bundle L and a holomorphic map

$$\pi : X \to C$$

to a nonsingular complete curve. The triple (X, L, π) is a *family of quasi-polarized* $K3$ *surfaces of degree* l if the fibers (X_ξ, L_ξ) are quasi-polarized $K3$ surfaces satisfying

$$\int_{X_\xi} L_\xi^2 = l$$

for every $\xi \in C$. The family (X, L, π) yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_l,$$

to the moduli space of quasi-polarized $K3$ surfaces of degree l.

We will often refer to the triple (X, L, π) just by π. Associated to π is the projective variety $\widetilde{X}$ obtained from the relative quasi-polarization,

$$X \to \widetilde{X} \subset \mathbb{P}(R^0\pi_*(L^n)^*) \to C,$$

for sufficiently large n. The complex manifold X may be a non-projective small resolution of the singular projective variety $\widetilde{X}$.

1.3. **Period domain.** Let V be a rank 22 integer lattice with intersection form $\langle , \rangle$ obtained from the second homology of a $K3$ surface,

$$V \stackrel{\sim}{=} U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

A 1-dimensional subspace $\mathbb{C} \cdot \omega \in V \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying

$$\langle \omega, \omega \rangle = 0 \ \text{ and } \ \langle \omega, \overline{\omega} \rangle > 0 \tag{5}$$

determines a Hodge structure of type $(1, 20, 1)$ on V,

$$V \otimes_{\mathbb{Z}} \mathbb{C} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2} = \mathbb{C} \cdot \omega \ \oplus \ (\mathbb{C} \cdot \omega \oplus \mathbb{C} \cdot \overline{\omega})^{\perp} \ \oplus \ \mathbb{C} \cdot \overline{\omega}.$$

Conversely, a Hodge structure of type $(1, 20, 1)$ determines a 1-dimensional subspace $\mathbb{C} \cdot \omega$ satisfying (5).

The moduli space M^V of Hodge structures of type $(1, 20, 1)$ on V is therefore an analytic open set of the 20-dimensional nonsingular isotropic quadric Q,

$$M^V \subset Q \subset \mathbb{P}(V \otimes_{\mathbb{Z}} \mathbb{C}).$$

The moduli space M^V is the *period domain*.

For nonzero $\beta \in V$, let $D^V_\beta \subset M^V$ denote the locus of Hodge structures for which $\beta \in V^{1,1}$. Certainly,

$$D^V_\beta = M^V \cap \beta^\perp \subset \mathbb{P}(V \otimes_{\mathbb{Z}} \mathbb{C})$$

where $\beta^\perp$ is the linear space orthogonal to β. Hence, D^V_β is simply a 19-dimensional hyperplane section of M^V.

1.4. **Local systems.** Let (X, L, π) be a quasi-polarized family of $K3$ surfaces over a nonsingular curve C. Let

$$\mathcal{V} = R^2\pi_*(\mathbb{Z}) \to C$$

denote the rank 22 local system determined by the middle cohomology of the fibration

$$\pi : X \to C.$$

The local system $\mathcal{V}$ is equipped with the fiberwise intersection form $\langle , \rangle$.

Let $\mathcal{M}^{\mathcal{V}}$ be the π-relative moduli space of Hodge structures

$$\mu : \mathcal{M}^{\mathcal{V}} \to C$$

with fiber

$$\mu^{-1}(\xi) = M^{\mathcal{V}_\xi}.$$

The moduli space $\mathcal{M}^{\mathcal{V}}$ is a complex manifold, and μ is a locally trivial fibration in the analytic topology.

Duality and homological push-forward yield a canonical map

$$\epsilon : \mathcal{V} \to H_2(X, \mathbb{Z})$$

where the right side can be viewed as a trivial local system. Let $H_2(X, \mathbb{Z})^\pi$ denote the kernel of the projective map

$$\pi_* : H_2(X, \mathbb{Z}) \to H_2(C, \mathbb{Z}).$$

For $h \in \mathbb{Z}$ and $\gamma \in H_2(X, \mathbb{Z})^\pi$, we will define a Noether-Lefschetz number $NL^\pi_{h,\gamma}$ for the $K3$ fibration π.

Informally, $NL^\pi_{h,\gamma}$ counts the number of point $\xi \in C$ for which there exists an integral class $\beta \in V_\xi$ of type $(1, 1)$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \ \text{ and } \ \epsilon(\beta) = \gamma.$$

The formal definition is given in Section 1.5.

1.5. **Classical intersection.** Define the relative divisor

$$\mathcal{D}^{\mathcal{V}}_{h,\gamma} \subset \mathcal{M}^{\mathcal{V}}$$

by the set of Hodge structures which contain a class $\beta \in \mathcal{V}_\xi$ of type $(1,1)$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \text{ and } \epsilon(\beta) = \gamma.$$

When $\mathcal{M}^{\mathcal{V}}$ is trivialized[4] over a Euclidean open set $U \subset C$,

$$\mathcal{M}^{\mathcal{V}_U} = M^V \times U,$$

the subset $\mathcal{D}^{\mathcal{V}}_{h,\gamma}$ restricts to

$$\mathcal{D}^{\mathcal{V}_U}_{h,\gamma} = \cup_\beta \; D^V_\beta \times U$$

where the union is over all $\beta \in V$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \text{ and } \epsilon(\beta) = \gamma.$$

Hence, $\mathcal{D}^{\mathcal{V}}_{h,\gamma} \subset \mathcal{M}^{\mathcal{V}}$ is a countable union of divisors.

The Noether-Lefschetz number is defined by a tautological intersection product. The family π determines a canonical section

$$\sigma : C \to \mathcal{M}^{\mathcal{V}}.$$

where

$$\sigma(\xi) = [H^{2,0}(X_\xi, \mathbb{C})] \in \mathcal{M}^{\mathcal{V}_\xi}$$

is the Hodge structure determined by the $K3$ surface X_ξ. Let

$$NL^\pi_{h,\gamma} = \int_C \sigma^*[\mathcal{D}^{\mathcal{V}}_{h,\gamma}]. \tag{6}$$

The divisor $\mathcal{D}^{\mathcal{V}}_{h,\gamma}$ may have infinitely many components. However, by the finiteness result of Proposition 1, $NL^\pi_{h,\gamma}$ is well-defined.

While $NL^\pi_{h,\gamma}$ is a classical intersection number, an excess calculation is required in case $\sigma(C) \subset \mathcal{D}^{\mathcal{V}}_{h,\gamma}$. The informal counting interpretation is not always well-defined.

Proposition 1. *$NL^\pi_{h,\gamma}$ is finite.*

Proof. Let L be the quasi-polarization on X. If there exists a point $\xi \in C$ for which L_ξ is ample, then L is π-relatively ample over an open set of C. If L_ξ is never ample, then the morphism

$$X \to \widetilde{X} \subset \mathbb{P}(R^0\pi_*(L^n))$$

[4]We take trivializations obtained from trivializing $R^2\pi_*(\mathbb{Z})$ compatibly with ϵ .

for sufficiently large n contracts divisors on X which intersect the generic fiber X_ξ in (-2)-curves. After modification[5] of L by these contracted divisors, a new quasi-polarization L' of X may be obtained which is π-relatively ample over a nonempty open set of C.

We assume now (after possible modification) the quasi-polarization L is π-relatively ample over a nonempty open set $U \subset C$. Let

$$d = \int_\gamma L$$

be the degree of γ. Let

$$l = \int_{X_\xi} L_\xi^2 > 0$$

be the degree of the $K3$ fibers of π.

Let $\beta \in \mathcal{V}_\xi$ of type $(1,1)$ satisfy

$$\langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \epsilon(\beta) = \gamma.$$

We will prove

$$\sigma(C) \subset \mathcal{M}^{\mathcal{V}}$$

intersects only finitely many components of $\mathcal{D}^{\mathcal{V}}_{h,\gamma}$.

Let k be an integer satisfying

$$d + lk > 0 \quad \text{and} \quad lk^2 + 2dk + 2h - 2 > -4.$$

The first step is to show

$$\tilde{\beta} = \beta + kc_1(L_\xi)$$

is an effective curve class on X_ξ by Riemann-Roch.

Let $L_{\tilde{\beta}}$ denote the unique line bundle on X_ξ with

$$c_1(L_{\tilde{\beta}}) = \tilde{\beta}.$$

By Serre duality,

$$H^2(X_\xi, L_{\tilde{\beta}}) = H^0(X_\xi, L^*_{\tilde{\beta}})^*$$

Since

$$\langle c_1(L^*_{\tilde{\beta}}), L_\xi \rangle \leq -d - lk < 0,$$

[5] A base change of $\pi : X \to C$ is not required since the modification can be averaged over the symmetries of the (-2)-curve configuration.

$h^0(X_\xi, L^*_{\tilde\beta})$ vanishes. Then, by Riemann-Roch,

$$\begin{aligned} h^0(X_\xi, L_{\tilde\beta}) &\geq \chi(X_\xi, L_{\tilde\beta}) - h^2(X_\xi, L_{\tilde\beta}) \\ &= \chi(X_\xi, L_{\tilde\beta}) \\ &= \frac{1}{2}\langle \tilde\beta, \tilde\beta\rangle + 2 \\ &> 0. \end{aligned}$$

Hence, $\tilde\beta$ is an effective curve class on X_ξ.

Consider first the open set $U \subset C$ over which L is π-relatively ample. Let

$$\mathcal{H} \to U$$

be the π-relative Hilbert scheme parameterizing of curves in $X_{\xi\in U}$ of degree

$$\langle \tilde\beta, c_1(L_\xi)\rangle = d + lk$$

and Euler characteristic

$$\chi(X_\xi, \mathcal{O}_{X_\xi}) - \chi(X_\xi, L^*_{\tilde\beta}) = -\frac{1}{2}\langle \tilde\beta, \tilde\beta\rangle = -\frac{1}{2}(lk^2 + 2dk + 2h - 2).$$

The scheme $\mathcal{H}$ is projective over U and of finite type.

An irreducible component $\mathcal{H}_{irr} \subset \mathcal{H}$ either dominates U or maps to a point $\xi \in U$. In the former case, the classes of curves represented by $\mathcal{H}_{irr}$ yield a *finite* monodromy invariant subset of $\mathcal{V}$. In the latter case, the curves represented by $\mathcal{H}_{irr}$ yield a single element of $\mathcal{V}_\xi$.

After shifting the finiteness statements back by $kc_1(L_\xi)$, we obtain the finiteness of the intersection geometry

$$\sigma(C) \cap \mathcal{D}^{\mathcal{V}}_{h,\gamma} \tag{7}$$

over $U \subset C$. Indeed, the dominant components H_{irr} correspond to finitely many excess intersections and the non-dominant components correspond to finitely many true intersections.

Finally consider the complement $U^c \subset C$. The complement is a finite set. For each $\xi^c \in U^c$, let $L^c_{\xi^c}$ be an ample line bundle. The above arugment using the ample bundles $L^c_{\xi^c}$ for the fibers X_{ξ^c} shows there are finitely many intersections in (7) over $U^c \subset C$ as well.

We conclude the intersection geometry is finite over all of C and the product

$$NL^\pi_{h,\gamma} = \int_C \sigma^*[\mathcal{D}^{\mathcal{V}}_{h,\gamma}]$$

is well-defined. □

Let γ_L denote the push-forward of the ample class on the fibers,

$$\gamma_L = c_1(L) \cap [X_\xi] \in H_2(X, \mathbb{Z})^\pi.$$

By an elementary comparison,

$$\sigma^*[\mathcal{D}^{\mathcal{V}}_{h,\gamma}] = \sigma^*[\mathcal{D}^{\mathcal{V}}_{h+d+\frac{l}{2},\gamma+\gamma_L}].$$

We obtain the following result.

Proposition 2. $NL^{\pi}_{h,\gamma} = NL^{\pi}_{h+d+\frac{l}{2},\gamma+\gamma_L}$.

The proof of Proposition 1 show the vanishing of the Noether-Lefschetz number for high h.

Proposition 3. *For fixed γ, the numbers $NL^{\pi}_{h,\gamma}$ vanish for sufficiently high h.*

The Noether-Lefschetz numbers $NL_{h,\gamma}(\pi)$ have non-trivial dependence on γ despite the linear equivalence

$$D^V_{\beta} \cong D^V_{\beta'}$$

on M^V. The Noether-Lefschetz numbers involve also the twisting of the local system $\mathcal{V}$ over C.

1.6. **Refinements.** The Noether-Lefschetz numbers $NL^{\pi}_{h,d}$ defined in Section 0.3 are obtained from the relation

$$NL^{\pi}_{h,d} = \sum_{\int_{\gamma} L=d} NL^{\pi}_{h,\gamma}. \tag{8}$$

The finiteness of the sum on the right is a consequence of the negative definiteness of the intersection matrix of divisors in X_{ξ} contracted by L_{ξ}. The invariants $NL^{\pi}_{h,\gamma}$ may be viewed as a refinement of $NL^{\pi}_{h,d}$ with the nonvanishing discriminant hypothesis lifted.

Further refined Noether-Lefschetz numbers may be defined with respect to any additional monodromy invariant data. For example, the divisibility m of an element $\beta \in \mathcal{V}_{\xi}$ is a monodromy invariant. Let

$$\mathcal{D}^{\mathcal{V}}_{m,h,\gamma} \subset \mathcal{M}^{\mathcal{V}}$$

be the divisor of Hodge structures which contain a class $\beta \in \mathcal{V}_{\xi}$ of type $(1,1)$ of divisibility m satisfying

$$\langle \beta, \beta \rangle = 2h - 2 \quad \text{and} \quad \epsilon(\beta) = \gamma.$$

We define

$$NL^{\pi}_{m,h,\gamma} = \int_C \sigma^*[\mathcal{D}_{m,h,\gamma}].$$

The relation

$$NL^{\pi}_{h,\gamma} = \sum_{m \geq 1} NL^{\pi}_{m,h,\gamma} \tag{9}$$

certainly holds.

1.7. **Intersection theory of $\mathcal{M}_l$.** Let $v \in V$ be a vector of norm l, and let

$$\mathcal{M}_v^V = v^\perp \cap \mathcal{M}^V.$$

Let Γ denote the group of orthogonal transformations of the lattice V, and let

$$\Gamma_v \subset \Gamma$$

be the subgroup fixing v. The moduli space of quasi-polarized $K3$ surfaces of degree l is the quotient

$$\mathcal{M}_l = \mathcal{M}_v^V / \Gamma_v.$$

The moduli space is a nonsingular orbifold. We refer the reader to [12] for a more detailed discussion.

In case $\triangle_l(h,d) \neq 0$, the above construction of $\mathcal{M}_l$ shows the definitions of the Noether-Lefschetz number by (3) and (8) agree.

2. Gromov-Witten theory

2.1. **BPS states for 3-folds.** Let (X, L, π) be a quasi-polarized family of $K3$ surfaces. While X may not be a projective variety, X carries a $(1,1)$-form ω_K which is Kähler on the $K3$ fibers of π. The existence of a fiberwise Kähler form is sufficient to define Gromov-Witten theory for vertical classes

$$0 \neq \gamma \in H_2(X, \mathbb{Z})^\pi.$$

The fiberwise Kähler form ω_K is obtained by a small perturbation of the quasi-Kähler form obtained from the quasi-polarization. The associated Gromov-Witten theory is independent of the perturbation used.[6]

Let $\overline{M}_g(X, \gamma)$ be the moduli space of stable maps from connected genus g curves to X. Gromov-Witten theory is defined by integration against the virtual class,

$$N^X_{g,\gamma} = \int_{[\overline{M}_g(X,\gamma)]^{vir}} 1. \tag{10}$$

The expected dimension of the moduli space is 0.

The Gromov-Witten potential $F^X(\lambda, v)$ for nonzero vertical classes is the series

$$F^X = \sum_{g \geq 0} \sum_{0 \neq \gamma \in H_2(X,\mathbb{Z})^\pi} N^X_{g,\gamma} \ \lambda^{2g-2} v^\gamma$$

[6]See [28, 34] for treatments of Gromov-Witten invariants for fiberwise Kähler geometry.

where λ and v are the genus and curve class variables. The BPS counts $n^X_{g,\gamma}$ of Gopakumar and Vafa are uniquely defined by the following equation:

$$F^X = \sum_{g\geq 0} \sum_{0\neq\gamma\in H_2(X,\mathbb{Z})^\pi} n^X_{g,\gamma}\, \lambda^{2g-2} \sum_{d>0} \frac{1}{d}\left(\frac{\sin(d\lambda/2)}{\lambda/2}\right)^{2g-2} v^{d\gamma}.$$

Conjecturally, the invariants $n^X_{g,\gamma}$ are integral and obtained from the cohomology of an as yet unspecified moduli space of sheaves on X.

2.2. **Reduced theory.** Let C be a connected, nodal, genus g curve. Let S be a $K3$ surface, and let $\beta \in \mathrm{Pic}(S)$ be a nonzero class. The moduli space $M_C(S,\beta)$ parameterizes maps from C to S of class β. Let

$$\nu : C \times M_C(S,\beta) \to M_C(S,\beta)$$

denote the projection, and let

$$f : C \times M_C(S,\beta) \to S$$

denote the universal map. The canonical morphism

$$R^\bullet\nu_*(f^*S)^\vee \to L^\bullet_{M_C} \tag{11}$$

determines a perfect obstruction theory on $M_C(S,\beta)$, see [2, 3, 32]. Here, $L^\bullet_{M_C}$ denotes the cotangent complex of $M_C(S,\beta)$.

Let Ω_S denote the cotangent bundle of S. Let Ω_ν and ω_ν denote respectively the sheaf of relative differentials of ν and the relative dualizing sheaf of ν. There are canonical maps

$$f^*(\Omega_S) \to \Omega_\nu \to \omega_\nu \tag{12}$$

The sections of the canonical bundle $H^0(S,K_S)$ determine a 1-dimensional space of holomorphic symplectic forms. Hence, there is a canonical isomorphism

$$T_S \otimes H^0(S,K_S) \stackrel{\sim}{=} \Omega_S$$

where T_S is the tangent bundle. We obtain a map

$$f^*(T_S) \to \omega_\nu \otimes (H^0(S,K_S))^\vee$$

and a map

$$R^\bullet\nu_*(\omega_\nu)^\vee \otimes H^0(S,K_S) \to R^\bullet\nu_*(f^*T_S)^\vee. \tag{13}$$

From (13), we obtain the cut-off map

$$\iota : \tau_{\leq -1}R^\bullet\nu_*(\omega_\nu)^\vee \otimes H^0(S,K_S) \to R^\bullet\nu_*(f^*T_S)^\vee.$$

The complex $\tau_{\leq -1}R^\bullet\nu_*(\omega_\nu)^\vee \otimes H^0(S,K_S)$ is represented by a trivial bundle of rank 1 tensored with $H^0(S,K_S)$ in degree -1. Consider the mapping cone $C(\iota)$ of ι. Certainly $R^\bullet\pi_*(f^*T_S)^\vee$ is represented by a

two term complex. An elementary argument using nonvanishing $\beta \neq 0$ shows the complex $C(\iota)$ is also two term.

By Ran's results[7] on deformation theory and the semiregularity map, there is a canonical map

$$C(\iota) \to L^{\bullet}_{M_C} \tag{14}$$

induced by (11), see [44]. Ran proves the obstructions to deforming maps from C to a holomorphic symplectic manifold lie in the kernel of the semiregularity map. After dualizing, Ran's result precisely shows (11) factors through the cone $C(\iota)$.

The map (14) defines a *new* perfect obstruction theory on $M_C(S,\beta)$. The conditions of cohomology isomorphism in degree 0 and the cohomology surjectivity in degree -1 are both induced from the perfect obstruction theory (11). We view (11) as the *standard* obstruction theory and (14) as the *reduced* obstruction theory.

Following [2, 3], the morphism (14) is an obstruction theory of maps to S relative to the Artin stack $\mathfrak{M}_g$ of genus g curves. A reduced absolute obstruction theory

$$E^{\bullet} \to L^{\bullet}_{\overline{M}_g(S,\beta)} \tag{15}$$

is obtained via a distinguished triangle in the usual way, see [2, 3, 32]. The obstruction theory (15) yields a reduced virtual class

$$[\overline{M}_g(S,\beta)]^{red} \in A_g(\overline{M}_g(S,\beta),\mathbb{Q})$$

of dimension g.

2.3. **BPS for $K3$ surfaces.** Let (S,ω_K) be a $K3$ surface with a Kähler form ω_K. Let $\beta \in \mathrm{Pic}(S)$ be a nonzero class of positive degree

$$\int_\beta \omega_K > 0.$$

We are interested in the following reduced Gromov-Witten integrals,

$$R_{g,\beta} = \int_{[\overline{M}_g(S,\beta)]^{red}} (-1)^g \lambda_g. \tag{16}$$

Here, the integrand λ_g is the top Chern class of the Hodge bundle

$$\mathbb{E}_g \to \overline{M}_g(S,\beta)$$

with fiber $H^0(C,\omega_C)$ over moduli point

$$[f : C \to S] \in \overline{M}_g(S,\beta).$$

[7]The required deformation theory can also be found in a recent paper by M. Manetti [35].

See [13, 19] for a discussion of Hodge classes in Gromov-Witten theory.

The definition of the BPS counts associated to the Hodge integrals (16) is straightforward. Let $\alpha \in \mathrm{Pic}(S)$ be a primitive class of positive degree with respect to ω_K. The Gromov-Witten potential $F_\alpha(\lambda, v)$ for classes proportional to α is

$$F_\alpha = \sum_{g\geq 0} \sum_{m>0} R_{g,m\alpha} \, \lambda^{2g-2} v^{m\alpha}.$$

The BPS counts $r_{g,m\alpha}$ are uniquely defined by the following equation:

$$F_\alpha = \sum_{g\geq 0} \sum_{m>0} r_{g,m\alpha} \, \lambda^{2g-2} \sum_{d>0} \frac{1}{d} \left(\frac{\sin(d\lambda/2)}{\lambda/2} \right)^{2g-2} v^{dm\alpha}.$$

We have defined BPS counts for both primitive and divisible classes.

The string theoretic calculations of Katz, Klemm and Vafa [22] via heterotic duality yield two conjectures.

Conjecture 1. *The BPS count $r_{g,\beta}$ depends upon β only through the square $\int_S \beta^2$.*

Assuming the validity of Conjecture 1, let $r_{g,h}$ denote the BPS count associated to a class β satisfying

$$\int_S \beta^2 = 2h - 2.$$

Conjecture 1 is rather surprising from the point of view of Gromov-Witten theory. By deformation arguments, the invariants $R_{g,\beta}$ depend upon both the divisibility m of β and $\int_S \beta^2$. Hence, BPS counts $r_{g,m,h}$ depending upon both the divisibility and the norm are well-defined unconditionally.

Conjecture 2. *The BPS counts $r_{g,h}$ are uniquely determined by the following equation:*

$$\sum_{g\geq 0} \sum_{h\geq 0} (-1)^g r_{g,h} (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{2g} q^h = \prod_{n\geq 1} \frac{1}{(1-q^n)^{20}(1-yq^n)^2(1-y^{-1}q^n)^2}.$$

As a consequence of Conjecture 2, $r_{g,h}$ vanishes if $g > h$ and

$$r_{g,g} = (-1)^g (g+1).$$

The first values are tabulated below:

$r_{g,h}$	$h=0$	1	2	3	4
$g=0$	1	24	324	3200	25650
1		-2	-54	-800	-8550
2			3	88	1401
3				-4	-126
4					5

The right side Conjecture 2 is related to the generating series of Hodge numbers of the Hilbert schemes of points $\text{Hilb}(S,n)$. The genus 0 specialization of Conjecture 2 recovers the Yau-Zaslow formula

$$\sum_{h\geq 0} r_{0,h} q^h = \prod_{n\geq 1} \frac{1}{(1-q^n)^{24}}$$

related to the Euler characteristics of $\text{Hilb}(S,n)$.

The Conjectures are proven in very few cases. A mathematical approach to the genus 0 Yau-Zaslow formula following [47] can be found in [4, 11, 14]. The Yau-Zaslow formula is proven for primitive classes β by Bryan and Leung [9]. If β has divisibility 2, the Yau-Zaslow formula is proven by Lee and Leung in [29]. Using Theorem 1, a complete proof of the Yau-Zaslow formula for all divisibilities is given in [25]. Since

$$R_{1,\beta} = \int_{[\overline{M}_1(S,\beta)]^{red}} -\lambda_1 = -\frac{\langle \beta,\beta\rangle}{24} R_{0,\beta},$$

we obtain

$$r_{1,h} = -\frac{h}{12}\ r_{0,h}$$

and Conjectures 1 and 2 from the genus 0 results.

Conjecture 2 for primitive classes β is connected to Euler characteristics of the moduli spaces of stable pairs on $K3$ by the correspondence of [42, 43]. A proof of Conjecture 2 for primitive classes is given in [38].

3. Theorem 1

3.1. **Result.** Consider a quasi-polarized family of $K3$ surfaces of degree l as in Section 1.2,

$$\pi : X \to C\ .$$

We restate Theorem 1 in terms of $\gamma \in H_2(X,\mathbb{Z})^\pi$ following the notation of Section 1.4.

Theorem 1. *For* $\gamma \neq 0$,

$$n^X_{g,\gamma} = \sum_h \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^\pi_{m,h,\gamma}.$$

3.2. **Proof.** Since the formulas relating the BPS counts to Gromov-Witten invariants are the same for X and the $K3$ surface, Theorem 1 is equivalent to the analogous Gromov-Witten statement:

$$N^X_{g,\gamma} = \sum_h \sum_{m=1}^{\infty} R_{g,m,h} \cdot NL^{\pi}_{m,h,\gamma} \tag{17}$$

for $\gamma \neq 0$.

Following the notation of Section 1.5, let σ denote the section

$$\sigma : C \to \mathcal{M}^{\mathcal{V}}$$

determined by the Hodge structure of the $K3$ fibers

$$\sigma(\xi) = [H^0(X, K_{X_\xi})] \in \mathcal{M}^{\mathcal{V}_\xi}.$$

For each $\xi \in C$, let

$$\mathcal{V}_\xi(m, h, \gamma) \subset \mathcal{V}_\xi$$

be the set of classes with divisibility m, square $2h-2$, and push-forward γ. Let

$$B_\xi(m, h, \gamma) = \{ \ \beta \in \mathcal{V}_\xi(m, h, \gamma) \mid \sigma(\xi) \in \beta^\perp \ \}.$$

By Proposition 1, the set $B_\xi(m, h, \gamma)$ is finite.

Equation (17) is proven by showing the contributions of the classes $B_\xi(m, h, \gamma)$ to both sides are the same. The set

$$B(m, h, \gamma) = \bigcup B_\xi(m, h, \gamma) \subset \mathcal{V}$$

can be divided into two disjoint subsets

$$B(m, h, \gamma) = B_{\text{iso}}(m, h, \gamma) \cup B_\infty(m, h, \gamma).$$

The elements of $B_{\text{iso}}(m, h, \gamma)$ are isolated while the the elements of $B_\infty(m, h, \gamma)$ form a finite local system over C,

$$\epsilon : B_\infty(m, h, \gamma) \to C. \tag{18}$$

We address the contributions of the isolated issues and the local system separately.

Consider first the local system (18). The contribution of ϵ to the Gromov-Witten invariant $N^X_{g,\gamma}$ is the integral

$$N^X_{g,\epsilon} = \int_{[\overline{M}_g(X,\epsilon)]^{vir}} 1$$

where $\overline{M}_g(X,\epsilon) \subset \overline{M}_g(X,\gamma)$ is the connected component[8] of the moduli space of stable maps which represent curve classes in ϵ. Alternatively,

$$N^X_{g,\epsilon} = \int_{[\overline{M}_g(\pi,\epsilon)]^{vir}} c_g(\mathbb{E}^*_g \otimes T_C) \tag{19}$$

where $\overline{M}_g(\pi,\epsilon) \subset \overline{M}_g(\pi,\gamma)$ is a connected component of the relative moduli space of maps. By standard arguments [13], the difference in the absolute and relative obstruction theories yields the Hodge integrand in (19).

The family π determines a canonical line bundle

$$K \to C$$

with fiber $H^0(X_\xi, K_{X_\xi})$ over $\xi \in C$. By the construction of the reduced class in Section 2.2,

$$[\overline{M}_g(\pi,\epsilon)]^{vir} = c_1(K^*) \cap [\overline{M}_g(\pi,\epsilon)]^{red}$$

where, on the right side, the reduced virtual class for the relative moduli space of maps appears. Expanding (19) yields

$$\begin{aligned} N^X_{g,\epsilon} &= \int_{[\overline{M}_g(\pi,\epsilon)]^{red}} c_g(\mathbb{E}^*_g \otimes T_C) \cdot c_1(K^*) \\ &= \int_{[\overline{M}_g(K3,m\alpha)]^{red}} (-1)^g \lambda_g \cdot \int_{B_\infty(m,h,\gamma)} c_1(K^*) \\ &= R_{g,m,h} \cdot \int_{B_\infty(m,h,\gamma)} c_1(K^*). \end{aligned}$$

In the second equality, α is primitive and satisfies

$$\langle m\alpha, m\alpha \rangle = 2h-2.$$

The contribution of the local system ϵ to the Noether-Lefschetz number $NL^\pi_{m,h,\gamma}$ is much easier to calculate. The local system represents an excess intersection contribution

$$\int_{B_\infty(m,h,\gamma)} c_1(\mathrm{Norm})$$

where Norm is the line bundle with fiber

$$\mathrm{Hom}(H^0(X_\xi, K_{X_\xi}), \mathbb{C}\cdot\beta)$$

[8]By connected component, we mean both open and closed. Formally, the condition is usually stated as a union of connected components.

at $\beta \in B_\infty(m,h,\gamma)$ lying over $\xi \in C$. Over $B_\infty(m,h,\gamma)$, the fibration $\mathbb{C} \cdot \beta$ is a trivial line bundle. Hence, the excess contribution of $B_\infty(m,h,\gamma)$ to $NL^\pi_{m,h,\gamma}$ is

$$\int_{B_\infty(m,h,\gamma)} c_1(K^*).$$

We conclude the contributions of $B_\infty(m,h,\gamma)$ to the left and right sides of equation (17) exactly match.

We consider now the contributions of the isolated classes $B_{iso}(m,h,\gamma)$ to the two sides of (17). Let

$$\beta \in B_{\text{ iso}}(m,h,\gamma)$$

be a isolated class lying over $\xi \in C$. We trivialize $\mathcal{M}^{\mathcal{V}}$ over a Euclidean open set $U \subset C$ as in Section 1.5. The local intersection of the section σ with the divisor

$$D^{V_\xi}_\beta \times U \subset M^{V_\xi} \times U$$

has an isolated point corresponding to (β, ξ). The local intersection multiplicity may not be 1. However, by deformation equivalence of the Gromov-Witten contributions on the left side of (17) and the intersection products on the right side of (17), we may assume *the local intersection multiplicity is 1* after local holomorphic perturbation of the section σ. Then, the contribution of the isolated class β to $NL^\pi_{m,h,\gamma}$ is certainly 1.

The final step is to show the contribution of the isolated class β with intersection multiplicity 1 to $N^X_{g,\gamma}$ is simply $R_{g,m,h}$. The result is obtained by a comparison of obstruction theories.

By the multiplicity 1 hypothesis, a connected component of the moduli space of stable maps to X coincides with the moduli stable of stable maps to fiber X_ξ,

$$\overline{M}_g(X_\xi, \beta) \subset \overline{M}_g(X,\gamma). \tag{20}$$

At the level of points, the assertion is obvious. The multiplicity 1 conditions prohibits any infinitesimal deformations of maps away from the fiber X_ξ and implies the scheme theoretic assertion.

From the fibration π, we obtain an exact sequence

$$0 \to T_{X_\xi} \to T_X|_{X_\xi} \to T_{C,\xi} \to 0, \tag{21}$$

and an induced map

$$\widetilde{\iota} \colon R^\bullet \nu_*(f^* T_{X_\xi})^\vee \to T^*_{C,\xi}$$

where the second complex is a trivial bundle in degree -1. Following the notation of Section 2.2, we have a canonical map

$$\iota : H^0(X_\xi, K_{X_\xi}) \to R^\bullet \nu_*(f^* T_{X_\xi})^\vee$$

where the first complex is a trivial bundle with fiber $H^0(X_\xi, K_{X_\xi})$ in degree -1. By Lemma 1 below, the composition

$$\widetilde{\iota} \circ \iota : H^0(X_\xi, K_{X_\xi}) \to T^*_{C,\xi}$$

is an isomorphism. Hence, by sequence (21), the obstruction theories $R^\bullet\nu_*(f^*T_X)^\vee$ and $C(\iota)$ differ by only by the Hodge bundle $\mathbb{E}_g \otimes T^*_{C,\xi}$. We conclude

$$[\overline{M}_g(X_\xi, \beta)]^{vir_X} = (-1)^g \lambda_g \cap [\overline{M}_g(X_\xi, \beta)]^{red}$$

where the virtual class on the left is obtained from the obstruction theory of maps to X via (20). The contribution of the isolated class β to $N^X_{g,\gamma}$ is thus $R_{g,h,m}$.

Since the contributions of $B_{\mathrm{iso}}(m,h,\gamma)$ to the left and right sides of equation (17) also match, the proof of Theorem 1 is complete. □

Lemma 1. *The composition*

$$\widetilde{\iota} \circ \iota : H^0(X_\xi, K_{X_\xi}) \to T^*_{C,\xi}$$

is an isomorphism.

Proof. Consider the differential of the period map at ξ,

$$T_{C,\xi} \to H^1(T_{X_\xi}) \to \mathrm{Hom}(H^0(K_{X_\xi}), H^1(\Omega_{X_\xi})).$$

The multiplicity 1 condition implies that the image of this map is not contained in the tangent space to the hyperplane $\beta^\perp = 0$. More explicitly, if we apply the cup-product pairing of $H^1(\Omega_{X_\xi})$ with the class $\beta \in H^2(X_\xi, \mathbb{Z})$, the composition

$$T_{C,\xi} \to H^0(K_{X_\xi})^* \otimes H^1(\Omega_{X_\xi}) \xrightarrow{\beta\cup} H^0(K_{X_\xi})^* \otimes \mathbb{C}$$

is nonzero. This sequence can be included in the diagram

$$\begin{array}{ccccccc}
T_{C_\xi} & \longrightarrow & H^1(T_{X_\xi}) & \longrightarrow & H^0(K_{X_\xi})^* \otimes H^1(\Omega_{X_\xi}) & \xrightarrow{\beta\cup} & H^0(K_{X_\xi})^* \\
\| & & \downarrow & & \downarrow & & \| \\
T_{C_\xi} & \longrightarrow & R^\bullet\nu_*(f^*T_{X_\xi}) & \longrightarrow & H^0(K_{X_\xi})^* \otimes R^\bullet\nu_*(f^*\Omega_{X_\xi}) & \longrightarrow & H^0(K_{X_\xi})^*
\end{array}$$

where the vertical maps are given by base-change morphisms and the bottom row is the map $(\widetilde{\iota} \circ \iota)^*$. Standard comparison results imply that this diagram commutes. Since the top row is nonvanishing, so is the bottom row. □

3.3. **Conjectures 1 and 2 revisited.** The proof of Conjectures 1 and 2 in the following case allows us to bound from below the h summation in Theorem 1.

Lemma 2. *If $\int_{K3} \beta^2 < 0$, then $r_{g,\beta} = 1$ if*

$$g = 0 \quad \text{and} \quad \int_{K3} \beta^2 = -2$$

and $r_{g,\beta} = 0$ otherwise.

Proof. Let S be a $K3$ surface, and let $\beta \in \mathrm{Pic}(S)$ be primitive with

$$\int_S \beta^2 = -2.$$

We may assume β is represented by an isolated -2 curve $P \subset S$. Let

$$\pi : X \to \triangle_0$$

be a 1-parameter deformation of S over the the disk $\triangle_0$ for which β fails (even infinitesimally) to remain algebraic. By the proof of Theorem 1, the reduced invariants $r_{g,m,\beta}$ are obtained[9] from the contribution of P to the BPS state counts of X. Since P is a rigid $(-1,-1)$ curve, P contributes a single BPS state [13]. We conclude

$$r_{g,m,\beta} = 1$$

if $(g,m) = (0,1)$ and $r_{g,m,\beta} = 0$ otherwise.

If $\beta \in \mathrm{Pic}(S)$ is primitive with square $2h-2$ strictly less than -2, then all reduced invariants $r_{g,m,\beta}$ vanish. The proof is obtained by considering elliptically fibered $K3$ surfaces $S \to \mathbb{P}^1$. Let

$$[s], [f] \in \mathrm{Pic}(S)$$

be the classes of a section and a fiber respectively. Then,

$$[s] + h[f], \ -[s] - h[f] \in \mathrm{Pic}(S)$$

are both primitive with square $2h-2$. Since the moduli spaces

$$\overline{M}_g\left(S, m([s]+h[f])\right), \ \overline{M}_g\left(S, m(-[s]-h[f])\right)$$

are easily seen to be empty, all reduced invariants $r_{g,m,\beta}$ vanish. $\square$

By Lemma 2, the integrals $r_{g,m,h<0}$ all vanish. Hence, Theorem 1 may be written as

$$n^X_{g,\gamma} = \sum_{h \geq 0} \sum_{m=1}^{\infty} r_{g,m,h} \cdot NL^\pi_{m,h,\gamma}.$$

[9]The local NL intersection number here is 1.

If Conjecture 1 and the vanishing $r_{g,h}$ for $g > h$ of Conjecture 2 hold, then

$$r_{g,h} = r_{g,m,h}$$

and Theorem 1 implies the following result. by relation (9).

Theorem 1*. *For $\gamma \neq 0$,*

$$n^X_{g,\gamma} = \sum_{h \geq g} r_{g,h} \cdot NL^{\pi}_{h,\gamma} \ .$$

The asterisk here indicates the dependence of Theorem 1* upon Conjectures 1 and 2.

3.4. **Invertibility.** Theorem 1* and Conjecture 2 imply the BPS states $n^X_{g,\gamma}$ of the total space contain exactly the same information as the Noether-Lefschetz numbers $NL^{\pi}_{h,\gamma}$.

Proposition 4*. *For $\gamma \in H_2(X,\mathbb{Z})^{\pi}$ of positive degree, the invariants $\{n_{g,\gamma}(\pi)\}_{g\geq 0}$ determine the Noether-Lefschetz numbers $\{NL_{h,\gamma}(\pi)\}_{h\geq 0}$ in terms of the invariants $\{r_{g,h}\}_{g,h\geq 0}$.*

Proof. Fix $\gamma \in H_2(X,\mathbb{Z})^{\pi}$. By Proposition 2, the numbers $NL_{h,\gamma}(\pi)$ vanish for $h > h_{top}$. So we need only determine

$$NL_{0,\gamma}, \ldots NL_{h_{top},\gamma}.$$

The equations

$$n_{g,\gamma}(\pi) = \sum_{h=g}^{h_{top}} r_{g,h} \cdot NL_{h,\gamma}(\pi)$$

for $g = 0, \ldots, h_{top}$ of Theorem 1* are triangular and invertible by Conjecture 2. □

4. Modular forms

4.1. **Overview.** We explain here Borcherds' work [6] relating Noether-Lefschetz numbers to Fourier coefficients of modular forms.[10] His results apply in great generality to arithmetic quotients of symmetric spaces associated to the orthogonal group $O(2,n)$ for any n. While we are mainly interested in the case of $O(2,19)$, we will first explain the statement in full generality. Other values of n play a role, for example,

[10]Borcherds' original result is modular only up to a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. The strengthening of [6] by the more recent rationality result of [39] removes the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ issue.

in studying 1-parameter families of $K3$ surfaces with generic Picard rank at least 2.

4.2. **Vector-valued modular forms of half-integral weight.** We first summarize standard facts and notation regarding modular forms of half-integral weight. In order to make sense of the modular transformation law with half-integer exponents, a double cover of the standard modular group $SL_2(\mathbb{Z})$ is required.

The metaplectic group $Mp_2(\mathbb{R})$ is the unique connected double cover $SL_2(\mathbb{R})$. The elements of $Mp_2(\mathbb{R})$ can be written in the form

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi(\tau) = \pm\sqrt{c\tau + d} \right)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\phi(\tau)$ is a choice of square root of the function $c\tau + d$ on the upper-half plane $\mathcal{H}$. The group structure is defined by the product

$$(A_1, \phi_1(\tau)) \cdot (A_2, \phi_2(\tau)) = (A_1 A_2, \phi_1(A_2\tau)\phi_2(\tau)) .$$

Here, we write $A\tau$ for the usual action of $SL_2(\mathbb{R})$ on $\tau \in \mathcal{H}$.

The group $Mp_2(\mathbb{Z})$ is the preimage of $SL_2(\mathbb{Z})$ under the projection map

$$\pi : Mp_2(\mathbb{R}) \to SL_2(\mathbb{R}).$$

It is generated by the two elements

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right),$$

where $\sqrt{\tau}$ denotes the choice of square root with positive real part.

Suppose we are given a representation ρ of $Mp_2(\mathbb{Z})$ on a finite-dimensional complex vector space V with the property that ρ factors through a finite quotient. Given $k \in \frac{1}{2}\mathbb{Z}$, we define a modular form of weight k and type ρ to be a holomorphic function

$$f : \mathcal{H} \to V$$

such that, for all $g = (A, \phi(\tau)) \in Mp_2(\mathbb{Z})$, we have

$$f(A\tau) = \phi(\tau)^{2k} \cdot \rho(g)(f(\tau)).$$

For $k \in \mathbb{Z}$ and ρ trivial, this reduces to the usual transformation rule.

If we fix an eigenbasis $\{v_\gamma\}$ for V with respect to T, we can take the Fourier expansion of each component of f at the cusp at infinity. That is, we write

$$f(\tau) = \sum_\gamma \sum_{k \in \mathbb{Z}} c_{k,\gamma} q^{k/R} v_\gamma \in V$$

where

$$q = e^{2\pi i \tau}$$

and R is the smallest positive integer for which $T^R \in \mathrm{Ker}(\rho)$. The function f is holomorphic at infinity if $c_{k,r} = 0$ for $k < 0$. The space $\mathrm{Mod}(Mp_2(\mathbb{Z}), k, \rho)$ of holomorphic modular forms of weight k and type ρ is finite-dimensional.

Given an integral lattice M with an even bilinear form $\langle , \rangle$ with signature $(2, n)$, we associate to M the following unitary representation of $Mp_2(\mathbb{Z})$. Let

$$M^\vee \subset M \otimes \mathbb{Q}$$

denote the dual lattice and $M^\vee/M$ the finite quotient. The pairing $\langle , \rangle$ extends linearly to a $\mathbb{Q}$-valued pairing on $M^\vee$. The functions $\frac{1}{2}\langle \gamma, \gamma \rangle$ and $\langle \gamma, \delta \rangle$ descend to $\mathbb{Q}/\mathbb{Z}$-valued functions on $M^\vee/M$.

We construct a representation ρ_M of $Mp_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[M^\vee/M]$. It suffices to define ρ_M in terms of the action of the generators T and S with respect to the standard basis v_γ for $\gamma \in M^\vee/M$,

$$\rho_M(T) v_\gamma = e^{2\pi i \frac{\langle \gamma, \gamma \rangle}{2}} v_\gamma \ ,$$

$$\rho_M(S) v_\gamma = \frac{\sqrt{i}^{\,n-2}}{\sqrt{|M^\vee/M|}} \sum_\delta e^{-2\pi i \langle \gamma, \delta \rangle} v_\delta \ .$$

Let N denote the smallest integer for which $N\langle \gamma, \gamma \rangle/2 \in \mathbb{Z}$ for all $\gamma \in M^\vee$. The representation factors through the finite index subgroup

$$\widetilde{\Gamma}(N) \subset Mp_2(\mathbb{Z})$$

consisting of elements (A, ϕ) for which

$$A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N.$$

We will be primarily interested in the dual representation ρ_M^* of $Mp_2(\mathbb{Z})$ on $\mathbb{C}[M^\vee/M]$. We have given the action of ρ_M to match Borcherds' notation.

4.3. **Heegner divisors.** Given the lattice M of type $(2, n)$ as before, consider the Hermitian symmetric domain

$$\mathcal{D} = \{\omega \in \mathbb{P}(M \otimes_{\mathbb{Z}} \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$$

naturally associated to M. We will study the quotient

$$\mathcal{X}_M = \mathcal{D}/\Gamma_M \tag{22}$$

of $\mathcal{D}$ by the arithmetic subgroup of $O(2, n)$

$$\Gamma_M = \{g \in \mathrm{Aut}(M) \mid g \text{ acts trivially on } M^\vee/M\}\,.$$

The quotient (22) is a quasi-projective algebraic variety.

For every $n \in \mathbb{Q}^{<0}$ and $\gamma \in M^\vee/M$, we associate a divisor class $y_{n,\gamma} \in \mathrm{Pic}(\mathcal{X}_M)$ as follows. Given an element $v \in M^\vee$, there is an associated hyperplane

$$v^\perp = \{\omega \in \mathcal{D} \mid \langle \omega, v\rangle = 0\}.$$

Both $\langle v, v\rangle$ and the residue class $v \bmod M$ are invariant under the action of Γ_M. Therefore, if we fix $n \in \mathbb{Q}$ and $\gamma \in M^\vee/M$, the set of $v \in M^\vee$ with

$$\langle v, v\rangle = n, \quad v \equiv \gamma \bmod M$$

is also Γ_M-invariant. The union over the set of the associated hyperplanes

$$\sum_{\substack{\langle v, v\rangle = n \\ v \equiv \gamma \bmod M}} v^\perp$$

is Γ_M-invariant and descends to an algebraic divisor

$$y_{n,\gamma} = \left(\sum_{\langle v,v\rangle = n,\ v\equiv\gamma \bmod M} v^\perp \right) / \Gamma_M.$$

The $y_{n,\gamma}$ are the *Heegner divisors* of $\mathcal{X}_M$. Because of the symmetry $v^\perp = (-v)^\perp$, there is a redundancy

$$y_{n,\gamma} = y_{n,-\gamma}$$

in our notation, and $y_{n,\gamma}$ is multiplicity 2 everywhere if $2\gamma \equiv 0 \bmod M$.

In the degenerate case where $n = 0$, we have the following prescription. The line bundle $\mathcal{O}(-1)$ on $\mathcal{D} \subset \mathbb{P}(M \otimes_\mathbb{Z} \mathbb{C})$ admits a natural Γ_M action and therefore descends to a line bundle K on $\mathcal{X}_M$. If $n = 0$ and $\gamma = 0$, we set

$$y_{0,0} = K^*.$$

If $n = 0$ and $\gamma \neq 0$, we set $y_{n,\gamma} = 0$.

We place the Heegner divisors in a formal power series $\Phi_M(q)$ with coefficients in $\mathrm{Pic}(\mathcal{X}_M) \otimes \mathbb{C}[M^\vee/M]$. More precisely, we consider the generating function

$$\Phi(q) = \sum_{n\in\mathbb{Q}^{\geq 0}} \sum_{\gamma\in M^\vee/M} y_{-n,\gamma} q^n v_\gamma \in \mathrm{Pic}(\mathcal{X}_M)[[q^{1/N}]] \otimes_\mathbb{Z} \mathbb{C}[M^\vee/M].$$

The main result of [6] together with the refinement of [39] yield the following Theorem.

Theorem ([6],[39]) *Let M have signature $(2,n)$. The generating function $\Phi(q)$ is an element of*

$$\mathrm{Pic}(\mathcal{X}_M) \otimes_{\mathbb{Z}} \mathrm{Mod}(Mp_2(\mathbb{Z}), 1+\frac{n}{2}, \rho_M^*).$$

As a consequence, given any linear functional

$$\lambda : \mathrm{Pic}(\mathcal{X}_M) \otimes \mathbb{C} \to \mathbb{C},$$

the contraction $\lambda(\Phi_M(q))$ is the Fourier expansion of a vector-valued modular form of weight $1+\frac{n}{2}$ and type ρ_M^*.

Borcherds' proof uses the singular theta lift of [5] to construct automorphic forms on $\mathcal{X}_M$ starting from vector-valued meromorphic modular forms on the upper half-plane. The zeroes and poles of these automorphic forms lie precisely along the Heegner divisors with multiplicity determined by the singular part of the initial modular form. Each such lifting gives a relation in $\mathrm{Pic}(\mathcal{X}_M)$. The total collection of relations arising in this way are encoded in the modularity statement.

In [5], Borcherds only shows that $\Phi_M(q)$ lies in a certain Galois closure of the space of modular forms. For the representations ρ arising in [5], MacGraw proves in [39] that $\mathrm{Mod}(Mp_2(\mathbb{Z}), k, \rho)$ admits a basis with rational coefficients. Therefore, the Galois closure does not enlarge the space.

4.4. **Application to K3 surfaces.** Let V be the rank 22 lattice obtained from the second cohomology of a $K3$ surface with fixed polarization L of norm l. In order to apply Borcherds' results to the moduli spaces $\mathcal{M}_l$, we consider the lattice of signature $(2,19)$

$$M = L^{\perp} = \{v \in V \mid \langle L, v\rangle = 0\}.$$

A direct check yields

$$M \cong \mathbb{Z}w \oplus U^2 \oplus E_8(-1)^2$$

where $\langle w, w\rangle = -l$. Therefore

$$M^{\vee}/M = \mathbb{Z}/l\mathbb{Z}$$

and is generated by $\frac{1}{l}w$. Here, we will write ρ_l for the representation ρ_M.

From the definitions, we find $\mathrm{Aut}(V,L) = \Gamma_M$, so we have the identification

$$\mathcal{M}_l = \mathcal{X}_M.$$

We claim the Heegner divisors correspond precisely to our Noether-Lefschetz divisors.

Lemma 3. *We have* $D_{h,d} = y_{n,\gamma}$*, where*

$$n = -\frac{\Delta_l(h,d)}{2l} \quad \textit{and} \quad \gamma \equiv d(\frac{1}{l}w) \bmod M.$$

Proof. The Noether-Lefschetz divisor $D_{h,d}$ is the quotient by Γ_M of the union of hyperplanes

$$\sum_{\substack{\langle\beta,\beta\rangle = 2h-2 \\ \langle L,\beta\rangle = d}} \beta^\perp.$$

It therefore suffices to establish a bijection between the two sets of hyperplanes. Given an element $\beta \in V$ satisfying

$$\langle\beta,\beta\rangle = 2h-2, \quad \langle\beta, L\rangle = d,$$

let $v = \beta - \frac{d}{l}L \in M \otimes_{\mathbb{Z}} \mathbb{Q}$ be the projection of β to $M = L^\perp$. A direct calculation shows

$$\frac{1}{2}\langle v,v\rangle = h - 1 - \frac{d^2}{2l} = \frac{\triangle_l(h,d)}{2l} \ ,$$
$$v \equiv d \cdot (\frac{1}{l}w) \bmod M \ .$$

Conversely, given $v \in M^\vee$ satisfying the above conditions,

$$\beta = v + \frac{d}{l}L$$

gives the inverse construction. Since $\beta^\perp = v^\perp$, we obtain the result. $\square$

It is important for our applications that the constant term $y_{0,0}$ of $\Phi_M(q)$ matches with the line bundle K^* from our excess calculation in the proof of Theorem 1. This occurs because automorphic forms can be viewed as sections of powers of K^* on $\mathcal{M}_l$.

Let π be a 1-parameter family of quasi-polarized $K3$ surfaces of degree l, and let ι be the associated morphism to moduli space:

$$\pi : X \to C,$$

$$\iota : C \to \mathcal{M}_l.$$

We can apply Borcherds' theorem to the functional on $\mathrm{Pic}(\mathcal{M}_l)$ given by

$$D \mapsto \int_C \iota^* D.$$

Corollary 3. *There is a vector-valued modular form of weight* $21/2$ *and type* ρ_l^*,

$$\Phi^\pi(q) = \sum_{r=0}^{l-1} \Phi_r^\pi(q) v_r \in \mathbb{C}[[q^{1/2l}]] \otimes \mathbb{C}[\mathbb{Z}/l\mathbb{Z}],$$

with nonzero coefficients determined by the equality

$$NL_{h,d}^\pi = \Phi_r^\pi \left[\frac{\triangle_l(h,d)}{2l} \right]$$

where $r \equiv d \bmod l$.

4.5. **Quartic $K3$ surfaces.** We now apply Borcherds' modularity to the study of $K3$ surfaces of degree 4. If $l = 4$, the isomorphism class of a rank two lattice $(\mathbb{L}, v)$ with primitive polarization $\langle v, v \rangle = l$ is determined only by the discriminant $\triangle$.

Given a 1-parameter family $\pi : X \to C$ of quasi-polarized $K3$ surfaces of degree 4, we have the generating function

$$\Phi^\pi(q) = \Phi_0^\pi(q) v_0 + \Phi_1^\pi(q) v_1 + \Phi_2^\pi(q) v_2 + \Phi_3^\pi(q) v_3$$

which is a modular form of weight $21/2$ and type ρ_4^* by Corollary 3.

Consider the scalar-valued power series

$$\phi^\pi(q) = \Phi_0^\pi(q) + \frac{1}{2}\Phi_1^\pi(q) + \Phi_2^\pi(q) + \frac{1}{2}\Phi_3^\pi(q).$$

By chasing definitions, we see $\phi^\pi(q)$ has the following property:

$$NL_{h,d}^\pi = \phi^\pi \left[\frac{\triangle_4(h,d)}{8} \right]. \tag{23}$$

The factor of $1/2$ is included to correct for the redundancy

$$\Phi_1^\pi(q) = \Phi_3^\pi(q).$$

Proposition 5. *The function* $\phi^\pi(q)$ *is a homogeneous polynomial of degree* 21 *in*

$$A = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{8}} \quad \text{and} \quad B = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{8}}.$$

Proof. While the vector $\Phi^\pi(q)$ is modular with respect to the full metaplectic group, $\phi^\pi(q)$ is a priori only modular with respect to the subgroup $\widetilde{\Gamma}(8) = \mathrm{Ker}(\rho_4^*)$. However, we can write $\phi^\pi(q)$ as a sum

$$\phi^\pi(q) = \frac{3}{4}\phi_+(q) + \frac{1}{4}\phi_-(q)$$

where

$$\phi_+(q) = \Phi_0^\pi(q) + \Phi_1^\pi(q) + \Phi_2^\pi(q) + \Phi_3^\pi(q),$$

$$\phi_-(q) = \Phi_0^\pi(q) - \Phi_1^\pi(q)) + \Phi_2^\pi(q) - \Phi_3^\pi(q).$$

Consider the congruence subgroup of $SL_2(\mathbb{Z})$

$$\Gamma^0(8) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \bmod 8 \right\}.$$

A direct calculation of the representation ρ_4^* shows that $\phi_+(q)$ and $\phi_-(q)$ are modular forms of weight $21/2$ with respect to

$$\widetilde{\Gamma}^0(8) = \left\{ (A, \phi) \in Mp_2(\mathbb{Z}) \mid A \in \Gamma^0(8) \right\}$$

and distinct characters

$$\chi_+, \chi_- : \widetilde{\Gamma}^0(8) \to \mathbb{C}^*.$$

Moreover, A and B are modular forms of weight $1/2$ with respect to $\widetilde{\Gamma}^0(8)$ and the same characters χ_+ and χ_- respectively.

We will not describe $\chi_\pm$ explicitly. While they are distinct, their squares are equal and $\chi = \chi_+^2 = \chi_-^2$ descends to a character

$$\chi : \Gamma^0(8) \to \mathbb{C}^*.$$

The character χ is specified completely by the following evaluations:

$$\chi(\Gamma^1(8)) = 1, \ \chi \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1, \ \chi \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix} = -1$$

where

$$\Gamma^1(8) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \bmod 8, a \equiv d \equiv 1 \bmod 8 \right\}.$$

Consider the space $\mathrm{Mod}(\Gamma^0(8), 11, \chi)$ of holomorphic modular forms of weight 11 and type χ. The space $\mathrm{Mod}(\Gamma^0(8), 11, \chi)$ is 12-dimensional space with basis

$$A^{22}, A^{20}B^2, \cdots, A^2B^{20}, B^{22}.$$

Both $\phi_+(q) \cdot A$ and $\phi_-(q) \cdot B$ lie in $\mathrm{Mod}(\Gamma^0(8), 11, \chi)$. Since A^{22}/B and B^{22}/A are not holomorphic at the boundary, we conclude $\phi_\pm(q)$ are each homogeneous polynomials of degree 21 in A and B and therefore so is $\phi^\pi(q)$. $\square$

5. Lefschetz pencil of quartics

5.1. **Quartics.** A general Lefschetz pencil of quartics can be viewed as a hypersurface of type $(4,1)$,

$$\pi : X_{4,1} \subset \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^1 \tag{24}$$

where the last projection is onto the second factor. Unfortunately, π contains 108 nodal fibers, so the family (24) does not fit the specifications of Section 1.2.

A family of quasi-polarized $K3$ surfaces of degree 4 can be obtained from the Lefschetz pencil π by the following construction. Let

$$\epsilon : C_{53} \xrightarrow{2-1} \mathbb{P}^1 \tag{25}$$

be the genus 53 hyperelliptic curve branched over the 108 points of $\mathbb{P}^1$ corresponding to the nodal fibers of π. The family

$$\epsilon^*(X_{4,1}) \to C_{53}$$

has 3-fold double point singularities over the 108 nodes of the fibers of the original family π. Let

$$\widetilde{\pi} : \widetilde{X} \to C_{53}$$

be obtained from a small resolution

$$\widetilde{X} \to \epsilon^*(X_{4,1}).$$

Then, $\widetilde{\pi}$ is easily seen to be a family of quasi-polarized $K3$ surfaces of degree 4. The quasi-polarization is the pull-back of $\mathcal{O}_{\mathbb{P}^3}(1)$.

5.2. **Invariants.** The Noether-Lefschetz numbers are defined in Section 1 only for the family $\widetilde{\pi}$. However, for convenience, we define

$$NL^{\pi}_{g,d} = \frac{1}{2} NL^{\widetilde{\pi}}_{g,d} \ .$$

Instead of a curve class γ, the degree d against the polarization is taken as the second subscript.

The family $\widetilde{\pi}$ may be viewed as twice the Lefschetz pencil of quartics. Let

$$\pi_{4,2} : X_{4,2} \subset \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^1$$

be the family obtained from a nonsingular Calabi-Yau hypersurface. The family $\pi_{4,2}$ may also be viewed as twice the Lefschetz pencil.

Lemma 4. $n^{\widetilde{X}}_{g,d} = n^{X_{4,2}}_{g,d}$.

Proof. It suffices to prove the analogous statement for Gromov-Witten invariants. Consider the degeneration of $X_{4,2}$ to the union

$$X_{4,1} \cup_{K3} X_{4,1}$$

of two $(4,1)$ hypersurfaces along a smooth $K3$ surface. The degeneration formula of [30, 31] implies

$$N_{g,d}^{X_{4,2}} = 2N_{g,d}^{X_{4,1}/K3}$$

where the latter term denotes the Gromov-Witten theory of $X_{4,1}$ relative to the $K3$ fiber. Since the Gromov-Witten theory of $K3 \times \mathbb{P}^1$ vanishes, the trivial degeneration

$$X_{4,1} \cup_{K3} (K3 \times \mathbb{P}^1)$$

yields the equality of relative and absolute invariants

$$N_{g,d}^{X_{4,1}} = N_{g,d}^{X_{4,1}/K3}.$$

To study the small resolution $\widetilde{\pi}$, consider the family of double covers

$$\epsilon_t : C_t \mapsto \mathbb{P}^1$$

ramified at 108 generic points which specializes to our particular double cover (25) as $t \to 0$. The behavior of Gromov-Witten theory in the conifold transition from

$$X_t = \epsilon_t^*(X_{4,1})$$

to $\tilde{X}$ has been calculated by Li and Ruan [30]:

$$N_{g,d}^{\widetilde{X}} = N_{g,d}^{X_t}.$$

By degenerating the base C_t to two copies of $\mathbb{P}^1$, we have a degeneration of X_t to two copies of $X_{4,1}$ attached at 54 smooth $K3$ fibers. As before, we apply the degeneration formula and the identification of relative and absolute invariants to obtain the equality

$$N_{g,d}^{\widetilde{X}} = N_{g,d}^{X_t} = 2N_{g,d}^{X_{4,1}} = N_{g,d}^{X_{4,2}}.$$

□

Instead of studying the Gromov-Witten invariants of $\widetilde{X}$, we may study the Gromov-Witten invariants of $X_{4,2}$.

5.3. Mirror symmetry.

5.3.1. *Overview.* The genus 0 invariants of $X_{4,2}$ are determined from hypergeometric series by the mirror transformation. The mirror formulas of Candelas, de la Ossa, Green, and Parkes [10] have been proven mathematically in many settings [15, 16, 33]. In particular, the case of $X_{4,2}$ is understood rigorously. We follow the notation of [41].

5.3.2. *Potential.* Let the variables T_1, T_2 correspond to the hyperplane classes

$$H_1 \subset \mathbb{P}^3, \quad H_2 \subset \mathbb{P}^1$$

respectively. The genus 0 potential of $X_{4,2}$ for classes restricted from $\mathbb{P}^3 \times \mathbb{P}^1$ is

$$\mathcal{F}(T_1, T_2) = \frac{1}{3}T_1^3 + 2T_1^2 T_2 + \sum_{d_1, d_2 \geq 0,\ (d_1, d_2) \neq (0,0)} N_{0,(d_1,d_2)}^{X_{4,2}}\ e^{d_1 T_1} e^{d_2 T_2}$$

where we follow the Gromov-Witten notation of Section 2. The curve class (d_1, d_2) is not a fiber class for $\pi^{4.2}$ if $d_2 > 0$.

5.3.3. *Hypergeometric series.* Let t_1, t_2 be new variables. Define the hypergeometric series $I_{i,j}(t_1, t_2)$ by

$$\sum_{i=0}^{3} \sum_{j=0}^{1} I_{i,j}(t_1, t_2) H_1^i H_2^j =$$

$$\sum_{d_1, d_2 \geq 0} e^{(H_1 + d_1)t_1} e^{(H_2 + d_2)t_2} \frac{\Pi_{r=0}^{4d_1 + 2d_2}(4H_1 + 2H_2 + r)}{\Pi_{r=1}^{d_1}(H_1 + r)^4\ \Pi_{r=1}^{d_2}(H_2 + r)^2}.$$

The right side, taken mod H_1^4 and H_2^2, is valued in $H^*(\mathbb{P}^3 \times \mathbb{P}^1, \mathbb{Q})$. Formally,

$$I_{i,j}(t_1, t_2) \in \mathbb{Q}[[t_1, e^{t_1}, t_2, e^{t_2}]].$$

The functions $I_{i,j}(t)$ form a solution of the Picard-Fuchs differential equation associated to the mirror geometry.

5.3.4. *Mirror transformation.* The mirror transformation is defined using two auxiliary functions. Let

$$F(e^{t_1}, e^{t_2}) = \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{\infty} e^{d_1 t_1} e^{d_2 t_2} \frac{(4d_1 + 2d_2)!}{(d_1!)^4 (d_2!)^2},$$

and let

$$G_{a,b}(e^{t_1}, e^{t_2}) = \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{\infty} e^{d_1 t_1} e^{d_2 t_2} \frac{(4d_1 + 2d_2)!}{(d_1!)^4 (d_2!)^2} \Big(\sum_{r=1}^{ad_1 + bd_2} \frac{1}{r} \Big)$$

for $a, b \geq 0$.

The mirror transformation relating the variables T_i and t_i is determined by the following equations:

$$T_1 = t_1 + \frac{4(G_{4,2}(e^{t_1}, e^{t_2}) - G_{1,0}(e^{t_1}, e^{t_2}))}{F(e^{t_1}, e^{t_2})},$$

$$T_2 = t_2 + \frac{2(G_{4,2}(e^{t_1}, e^{t_2}) - G_{0,1}(e^{t_1}, e^{t_2}))}{F(e^{t_1}, e^{t_2})}.$$

Exponentiation yields

$$e^{T_1} = e^{t_1} \cdot \exp\left(\frac{4(G_{4,2}(e^{t_1}, e^{t_2}) - G_{1,0}(e^{t_1}, e^{t_2}))}{F(e^{t_1}, e^{t_2})}\right),$$

$$e^{T_2} = e^{t_2} \cdot \exp\left(\frac{2(G_{4,2}(e^{t_1}, e^{t_2}) - G_{0,1}(e^{t_1}, e^{t_2}))}{F(e^{t_1}, e^{t_2})}\right).$$

Together, the above four equations define a change of variables from formal series in $T_1, e^{T_1}, T_2, e^{T_2}$ to formal series in $t_1, e^{t_1}, t_2, e^{t_2}$. The mirror transformation is easily seen to be invertible.

5.3.5. *Genus 0 invariants.* The genus 0 potential $\mathcal{F}$ is determined by mirror symmetry,

$$\mathcal{F}(T_1(t_1,t_2), T_2(t_1,t_2)) = \left(\frac{2I_{1,1} - I_{2,0}}{I_{1,0}}\right)\left(\frac{I_{3,0}}{I_{1,0}}\right) + 2\left(\frac{I_{2,0}}{I_{1,0}}\right)\left(\frac{I_{2,1}}{I_{1,0}}\right) - 2\left(\frac{I_{3,1}}{I_{1,0}}\right).$$

The arguments of the functions on the right side are understood to be t_1 and t_2. The genus 0 BPS states $n_{0,d}^{X_{4,2}}$ are determined by $\mathcal{F}$.

5.4. **Proof of Theorem 2.** Consider twice the Lefschetz pencil of quartics

$$\widetilde{\pi} : \widetilde{X} \to C_{53}.$$

Corollary 1 in genus 0 is

$$n_{0,d}^{\widetilde{X}} = \sum_{h=0}^{\infty} r_{0,h} \cdot NL_{h,d}^{\widetilde{\pi}} \,. \tag{26}$$

We now solve for the Noether-Lefschetz numbers of $\tilde{\pi}$. By (23),

$$NL_{h,d}^{\widetilde{\pi}} = \phi^{\widetilde{\pi}}\left[\frac{\triangle_4(h,d)}{8}\right]$$

where $\phi^{\widetilde{\pi}}(q)$ is a homogeneous polynomial of degree 21 in A and B. We need only 22 equations to determine $\phi^{\widetilde{\pi}}(q)$. Using the mirror symmetry calculation of $n_{0,d}^{\widetilde{X}}$, equation (26) provides infinitely many relations. In particular, $\phi^{\widetilde{\pi}}(q)$ is easily determined by linear algebra.

The precise formula for $\phi^{\widetilde{\pi}}$ is 2Θ where Θ is given in Section 0.6 since $\widetilde{\pi}$ is twice the Lefschetz pencil of quartics. The modular form Θ was first computed in [24].

5.5. **Modular identity.** Equation (26) may be viewed as a rather intricate relation between hypergeometric functions (after mirror transformation) on the left and modular forms on the right. Let

$$\mathcal{G}(q) = -\frac{2}{q} + 168 + \sum_{d\geq 1} n_{0,d}^{X_{4,2}} q^{\frac{d^2}{8}}$$

be the generating function determined by the property

$$\sum_{d=1}^{\infty}\sum_{k=1}^{\infty} n_{0,d}^{X_{4,2}} \frac{1}{k^3} e^{dkT_1} = \left(\mathcal{F}(T_1,T_2) - \frac{1}{3}T_1^3 - 2T_1^2 T_2\right)|_{e^{T_2}=0}$$

where $\mathcal{F}$ is determined as above.

Corollary 4. *We have the equality*

$$\mathcal{G}(q) = 2\frac{\Theta(q)}{\Delta(q)} \ ,$$

where $\Theta(q)$ *is given in Section 0.6 and*

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \ .$$

Such relations are produced by Theorem 1 for many classical examples. For any 1-parameter family of $K3$ surfaces obtained via a toric complete intersection, there is an associated identity of special functions. The relation obtained from the STU model studied in [25] is the Harvey-Moore identity. In fact, the Harvey-Moore identity is the *only* one for which a direct proof (avoiding Theorem 1) is known. The proof is due to Zagier and can be found in [25].

5.6. **Proof of Corollary 2.** Let π be the Lefschetz pencil of quartic $K3$ surfaces. The difference between $NL^{\pi}_{h,d}$ and the degree of

$$\overline{\mathcal{D}}_{h,d} \subset \mathbb{P}(\mathrm{Sym}^4(V^*))$$

is simply the contribution of the nodal quartics. The nodal quartics contribute to $NL^{\pi}_{h,d}$ but not the hypersurface $\overline{\mathcal{D}}_{h,d}$.

Using the relation $NL^{\pi}_{h,d} = \frac{1}{2}NL^{\widetilde{\pi}}_{h,d}$, we can study instead the doubled family. The Picard lattice of each of the 108 fibers of $\widetilde{\pi}$ corresponding to the original nodal fibers of π is

$$\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}. \tag{27}$$

We use here the genericity of the Lefschetz pencil π.

The equation $\langle \beta, L \rangle = d$ is solvable in the lattice (27) if and only if d is divisible by 4. Then, $\langle \beta, \beta \rangle = 2h - 2$ is solvable if and only if

$$4(\frac{d}{4})^2 - 2n^2 = 2h - 2$$

in which case there are two solutions. In the solvable cases,

$$\triangle_4(h, d) = 8n^2.$$

Hence, the contribution of the nodal fiber to the Noether-Lefschetz numbers of $\widetilde{\pi}$ is

$$\Psi(q) = 108 \cdot 2 \sum_{n>0} q^{n^2}.$$

The Corollary follows by halving. □

6. Direct Noether-Lefschetz calculations

6.1. **Overview.** We apply Corollary 3 to directly study $K3$ surfaces of low degree via a more sophisticated approach to modular forms. The key idea is to construct a basis of the space of vector-valued modular forms of Corollary 3 instead of working with the much larger space of scalar-valued modular forms as in Section 4.5. For many classical families, the dimensions of the associated spaces of vector-valued modular forms are very small. The Noether-Lefschetz numbers can often be specified by a few classical calculations. In particular, we see another derivation of Theorem 2.

6.2. **Rankin-Cohen brackets.** Since each component of a vector-valued modular form is a half-weight modular form of level $2l$, we can use a basis of the latter to construct all vector-valued modular forms. In practice, however, the method is tedius since the dimensions of the spaces of scalar-valued modular forms are much larger. We will instead apply the following shortcut for low degree $K3$ surfaces.

Let $f(q)$ and $g(q)$ be scalar-valued level N modular forms on the upper-half plane $\mathcal{H}$ of weights k_1 and k_2 respectively. For each integer $n \geq 0$, the n-th Rankin-Cohen bracket is a bilinear differential operator defined by the expression

$$[f(q), g(q)]_n = \sum_{r=0}^{n} (-1)^r \binom{n+k-1}{n-r} \binom{n+l-1}{r} f^{(r)}(q) \cdot g^{(n-r)}(q),$$

where $f^{(r)}$ denote r applications of the differential operator

$$\frac{d}{d\tau} = q\frac{d}{dq} \ .$$

For $n = 0$, the 0-th bracket is just multiplication.

The key feature of Rankin-Cohen brackets is the preservation of modularity. Suppose we are given a representation ρ of $Mp_2(\mathbb{Z})$ on V, a modular form $f \in \mathrm{Mod}(Mp_2(\mathbb{Z}), k_1, \rho)$ of weight k_1 and type ρ, and a scalar-valued modular form $g \in \mathrm{Mod}(SL_2(\mathbb{Z}), k_2)$ of weight k_2 and level 1. Let

$$f(q) = \sum_\gamma f_\gamma(q) v_\gamma \in V$$

denote the decomposition of f into components with respect to some basis of V. For each integer $n \geq 0$, the Rankin-Cohen bracket is a holomorphic function on $\mathcal{H}$ with values in V defined by

$$[f, g]_n(q) = \sum_\gamma [f_\gamma(q), g(q)]_n v_\gamma.$$

We then have the following result.

Lemma 5. *$[f, g]_n(q) \in \mathrm{Mod}(Mp_2(\mathbb{Z}), k_1 + k_2 + 2n, \rho)$.*

Proof. For scalar-valued modular forms, a proof is given in [48]. Since g is scalar-valued and level 1, the same argument translates to the vector-valued context without change. □

6.3. **Bases of modular forms.** Following the notation of Corollary 3, we now look for modular forms of weight 21/2 and type ρ_l^* for even

$$l = 2, 4, 6, 8\ .$$

From the dimension formula given in Section 7 below,

$$\dim(\mathrm{Mod}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)) = 2, 3, 4, 5$$

for $l = 2, 4, 6, 8$ respectively. We are only interested[11] in the subspace

$$\mathrm{Mod}_0(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)$$

of forms $\sum f_i(q) v_i$ where $f_r(q)$ is a cusp form for $r \neq 0$. In the $l = 8$ case, we have a 4-dimensional subspace.

We can use Rankin-Cohen brackets to construct explicit bases. Indeed, for each l, there is a canonical weight 1/2 modular form given by the Siegel theta function (see [5], Section 4),

$$\theta^{(l)}(q) = \sum_{i=0}^{l} \sum_s q^{\frac{(ls+i)^2}{2l}} v_i \in \mathrm{Mod}(Mp_2(\mathbb{Z}), 1/2, \rho_l^*).$$

Therefore, for $n = 0, 1, 2, 3$, Lemma 5 gives us a modular form,

$$F_n^l(q) = [\theta^{(l)}(q), E_{10-2n}(q)]_n \in \mathrm{Mod}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*),$$

[11] The cusp condition is obtained from Borcherds' results and was omitted in the statement of Corollary 3 for simplicity.

of weight $21/2$ where $E_{2k}(q)$ denotes Eisenstein series of weight $2k$.

Using the explicit formula for Rankin-Cohen brackets and the dimension formula, the following Lemma is obtained by calculating the initial Taylor coefficients.

Lemma 6. *For $l = 2, 4, 6$, the modular forms*

$$F_n^l(q) = [\theta^{(l)}(q), E_{10-2n}(q)]_n, n = 0, \dots, l/2$$

form a basis of $\mathrm{Mod}(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)$. *For $l = 8$, the modular forms for $n = 0, \dots 3$ form a basis of the subspace* $\mathrm{Mod}_0(Mp_2(\mathbb{Z}), 21/2, \rho_l^*)$.

6.4. **Classical families of $K3$ surfaces.** A general $K3$ surface of degree $l = 2, 4, 6, 8$ is either a branched cover of $\mathbb{P}^2$ (for $l = 2$) or a complete intersection in projective space. We obtain 1-parameter families of quasi-polarized $K3$ surfaces of degree l by taking a generic Lefschetz pencil of these constructions (and resolving singularities as discussed in Section 5.1). Because the space of vector-valued forms is of low dimension, we only need a few classical constraints to completely determine the associated modular form. In fact, we will use only the following constraints:

(i) the degree of the Hodge bundle $R^2\pi_*\mathcal{O}$ (the coefficient of $q^0 v_0$),
(ii) the number of nodal fibers (the coefficient of $q^1 v_0$),
(iii) vanishing obtained from Castelnuovo's bound in Lemma 7 below.

The following result is a special case of Castelnuovo's bound for projective curves [1].

Lemma 7. *Given a $K3$ surface with very ample bundle L and an primitive curve class β, we have the inequality*

$$\langle \beta, \beta \rangle \leq 2\binom{L \cdot \beta - 1}{2} - 2 \ .$$

We now apply these constraints for 1-parameter families of $K3$ given by Lefschetz pencils for $l = 2, 4, 5, 6$.

• *Degree 2 $K3$ surfaces*

A generic degree $K3$ surface of degree 2 is a double cover of $\mathbb{P}^2$ branched along a nonsingular sextic plane curve. Consider a family

$$R \subset \mathbb{P}^1 \times \mathbb{P}^2$$

of sextics defined by a generic hypersurface of type $(2,6)$. Let X be the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified over R. Since all the singular fibers of

$$R \to \mathbb{P}^1$$

are irreducible and nodal, the associated family of

$$\pi : X \to \mathbb{P}^1$$

of $K3$ surfaces has only 3-fold double-point singularities (which admit small resolutions).

The degree of the Hodge bundle is -1 by a Riemann-Roch calculation. The number of nodal fibers of π is 150, twice the degree of the discriminant locus of sextics. Since we have a 2-dimensional space of forms, the generating series of Noether-Lefschetz numbers is the vector-valued modular form

$$\overrightarrow{\Theta}(q) = -F_0^{(2)}(q) - \frac{1}{2}F_1^{(2)}(q).$$

In the case of $l = 2$, the discriminant Δ of a rank 2 lattice with degree 2 polarization determines the coset class δ by $\delta = \Delta \mod 2$. So there is no loss of information if we replace $\overrightarrow{\Theta}(q)$ by the sum of the components $\Theta(q) = \overrightarrow{\Theta}_0 + \overrightarrow{\Theta}_1$.

If we consider the theta functions

$$U = \sum_{n\in\mathbb{Z}} q^{n^2/4}, \quad V = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/4},$$

we can express Θ as a polynomial function of A and B:

$$\begin{aligned}\Theta(q) &= \frac{1}{1024}(U^{21} - 12U^{17}V^4 - 402U^{13}V^8 - 572U^9V^{12} - 39U^5V^{16} \\ &= -1 + 150q + 1248q^{5/4} + 108600q^2 + 332800q^{9/4} + 5113200q^3 \cdots .\end{aligned}$$

To see equivalence of the two expressions, we observe both are modular forms of weight $21/2$ with respect to $\Gamma(4)$ and check the agreement of sufficiently many coefficients.

- *Degree* 4 $K3$ *surfaces*

A generic $K3$ surface of degree 4 is a quartic hypersurface in $\mathbb{P}^3$. If we take a generic Lefschetz pencil of such quartics, the degree of the Hodge bundle is -1. Using Lemma 7, the Noether-Lefschetz degrees associated to the lattices

$$\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$$

both vanish. Indeed, by choosing a generic pencil, we can assume all fibers containing these Picard lattices have very ample quasi-polarization. The coefficients of $q^0 v_0, q^{1/8} v_1$, and $q^{1/2} v_2$ determine

$$\overrightarrow{\Theta}(q) = -F_0^{(4)}(q) - \frac{5}{4} F_1^{(4)}(q) - \frac{16}{21} F_2^{(4)}(q).$$

Again, as in the degree 2 case, we can recover all Noether-Lefschetz degrees from

$$\Theta(q) = \overrightarrow{\Theta}_0(q) + \overrightarrow{\Theta}_1(q) + \overrightarrow{\Theta}_2(q).$$

In terms of

$$A = \sum_{n \in \mathbb{Z}} q^{n^2/8}, \quad B = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/8},$$

we recover the expression for $\Theta(q)$ given in Section 0.6 since both are modular forms of weight $21/2$ and level 8 which agree on initial terms.

• *Degree* 6 $K3$ *surfaces*

A generic $K3$ surface of degree 6 is the intersection of a quadric and cubic hypersurface in $\mathbb{P}^4$. We have two basic families. We can fix a quadric and take a Lefschetz pencil of cubics or vice versa. In each case, we have vanishings associated to the lattices

$$\begin{pmatrix} 6 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix}$$

from the Castelnuovo bound. Along with the Hodge bundle degree and the number of nodal fibers, we completely determine the Noether-Lefschetz series.

For the first family, the Hodge and nodal degrees are -1 and 98 respectively. We obtain the series

$$\overrightarrow{\Theta}(q) = -F_0^{(6)}(q) - \frac{49}{24} F_1^{(6)}(q) - \frac{8}{3} F_2^{(6)}(q) - \frac{12}{5} F_3^{(6)}(q).$$

For the second family, the Hodge and nodal degrees are -1 and 7. We obtain the series

$$\overrightarrow{\Theta}(q) = -F_0^{(6)}(q) - \frac{17}{8} F_1^{(6)}(q) - \frac{22}{7} F_2^{(6)}(q) - \frac{18}{5} F_3^{(6)}(q).$$

One can read off other classical calculations from our results. For example, the number of surfaces containing elliptic plane curves or

containing lines are the Noether-Lefschetz degrees associated to the lattices

$$\begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 6 & 1 \\ 1 & -2 \end{pmatrix}$$

respectively. In the first family, the degrees are 0 and 168 respectively. In the second family, the degrees are 10 and 198. In both cases, the numbers agree with earlier enumerative calculations.

• *Degree* 8 $K3$ *surfaces*

A generic $K3$ surface of degree 8 is the intersection of three quadric hypersurfaces in $\mathbb{P}^5$. The basic family comes from fixing two quadrics and allowing the third to vary in a Lefschetz pncil. Again, the series is determined by the Hodge term, the nodal term, and the two Castelnuovo vanishings from Lemma 7. The Hodge term is given by -1, and the number of nodal fibers is 80. We find

$$\overrightarrow{\Theta}(q) = -F_0^{(8)}(q) - \frac{49}{18}F_1^{(8)}(q) - \frac{128}{27}F_2^{(8)}(q) - \frac{256}{45}F_3^{(8)}(q).$$

Again, we can read off that the number of fibers containing a line is 128, agreeing with the classical calculation.

For all the classical examples discussed above, the mirror symmetry calculation of the genus 0 Gromov-Witten invariants is solvable in terms of hypergeometric functions. In each case, Theorem 1 yields a remarkable identity with hypergeometric functions (after mirror transformation) on the left and modular forms on the right, as in Section 5.5.

7. Picard rank of $\mathcal{M}_l$

The Picard ranks of the moduli spaces of quasi-polarized $K3$ surfaces $\mathcal{M}_l$ are unknown. By an argument of O'Grady, the ranks can grow arbitrarily large [40]. Let

$$\mathrm{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} \subset \mathrm{Pic}(\mathcal{M}_l) \otimes \mathbb{Q} \tag{28}$$

denote the span of the Noether-Lefschetz divisors $D_{h,d}$. We make the following conjecture.

Conjecture 3. *The inclusion is an isomorphism,*

$$\mathrm{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} \cong \mathrm{Pic}(\mathcal{M}_l) \otimes \mathbb{Q}.$$

Bruinier has calculated the dimension of the space $\text{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q}$ in [7]. If Conjecture 3 holds, we obtain a formula for the Picard rank of $\mathcal{M}_l$.

We now recount Bruinier's formula for the span of the Noether-Lefschetz divisors. By Borcherds' work, we have a map

$$\text{Mod}(Mp_2(\mathbb{Z}), \rho_l^*, 21/2)^* \to \text{Pic}(\mathcal{M}_l) \otimes \mathbb{C}. \tag{29}$$

Let $\text{Cusp}(Mp_2(\mathbb{Z}), \rho_l^*, 21/2)$ denote the subspace of cusp forms — modular forms for which the Fourier coefficients $c_{0,\gamma}$ vanish for all γ. The map (29) induces a map

$$\text{Cusp}(Mp_2(\mathbb{Z}), \rho_l^*, 21/2)^* \to (\text{Pic}(\mathcal{M}_l) \otimes \mathbb{C})/\mathbb{C}K, \tag{30}$$

where K is the Hodge bundle on $\mathcal{M}_l$. Bruinier shows the map (30) is injective [7]. Specifically, if L is a $(2, n)$ lattice containing two copies of U as direct summands, Bruinier shows that every relation among Heegner divisors is obtained from Borcherds' theta lifting. Therefore,

$$\dim\ \text{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} = 1 + \dim\ \text{Cusp}(Mp_2(\mathbb{Z}), \rho_l^*, 21/2).$$

A direct calculation of the dimension of the space of cusp forms via Riemann-Roch yields the following evaluation [7]:

$$\begin{aligned} \dim\ \text{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} \quad = \quad & \frac{38}{24} + \frac{31}{24} l - \frac{1}{8\sqrt{l}} \text{Re}(G(2, 2l)) \\ & - \frac{1}{6\sqrt{3l}} \text{Re}(e^{-2\pi i \frac{19}{24}} (G(1, 2l) + G(-3, 2l))) \\ & - \sum_{k=0}^{l/2} \left\{ \frac{k^2}{2l} \right\} - C, \end{aligned}$$

where $G(a, b)$ denotes the quadratic Gauss sum

$$G(a, b) = \sum_{k=0}^{b-1} e^{2\pi i \frac{ak^2}{b}},$$

the braces $\{,\}$ denote fractional part, and C is the cardinality of the set

$$\left\{ k \ \middle|\ 0 \le k \le \frac{l}{2}, \frac{k^2}{2l} \in \mathbb{Z} \right\}.$$

For $l = 2, 4, 6$, the formula yields

$$\dim\ \text{Pic}(\mathcal{M}_l)^{NL} \otimes \mathbb{Q} = 2, 3, 4$$

respectively. For $l = 2$ and 4, we have agreement with the Picard ranks of $\mathcal{M}_l$ calculated in [23, 45, 46]. Hence, the inclusion (28) is an isomorphism in at least the first two cases.

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Department of Mathematics
Princeton University
Princeton, NJ 08544, USA
dmaulik@math.princeton.edu

Department of Mathematics
Princeton University
Princeton, NJ 08544, USA
rahulp@math.princeton.edu